Simplex in Matrix Notation

$$\boxed{\begin{aligned} \max \quad & \zeta(x) = c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}}$$

$x \in \mathbb{R}^n, \quad c \in \mathbb{R}^n$

$A^{m \times n}, \quad b \in \mathbb{R}^m$

$$\boxed{\begin{aligned} \max \quad & \zeta(x) = c^T x \\ \text{s.t.} \quad & w = b - Ax \\ & x, w \geq 0 \end{aligned}}$$

$w \in \mathbb{R}^m$

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$w \in \mathbb{R}^m$

simplex iterates

$$\boxed{\begin{aligned} \max\ & \zeta(x) = \bar{\zeta} + \bar{c}^T \bar{x} \\ \text{s.t.}\ & \bar{w} = \bar{b} - \bar{A}\bar{x} \\ & \bar{x}, \bar{w} \ge 0 \end{aligned}}$$

Simplex in Matrix Notation

$$\boxed{\begin{aligned} \max\ & \zeta(x) = c^T X \\ \text{s.t.}\ & AX + W = b \\ & X, W \geq 0 \end{aligned}}$$

$m \times (n+m)$ $\quad \mathbb{R}^{n+m}$

$$\begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} X \\ W \end{bmatrix} = b$$

Simplex iterates

$$\boxed{\begin{aligned} \max\ & \zeta(x) = \bar{\zeta} + \bar{c}^T \bar{X} \\ \text{s.t.}\ & \bar{A}\bar{X} + \bar{W} = \bar{b} \\ & \bar{X}, \bar{W} \geq 0 \end{aligned}}$$

$m \times (n+m)$ $\quad \mathbb{R}^{n+m}$

$$\begin{bmatrix} \bar{A} & I \end{bmatrix} \begin{bmatrix} \bar{X} \\ \bar{W} \end{bmatrix} = \bar{b}$$

Basic and Non-Basic Variables

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ w_1 \\ \vdots \\ w_m \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix} = \begin{pmatrix} X_N \\ \\ X_B \end{pmatrix}$$

simplex iterations

N - non-basic

B - basic

$$\underline{c^T X} = \sum_{j=1}^{n} c_j x_j = \underbrace{\sum_{j=1}^{n} c_j x_j}_{N} + \underbrace{\sum_{j=n+1}^{n+m} 0 \cdot x_j}_{B}$$

$$= \sum_{j \in B} c_j x_j + \sum_{j \in N} c_j x_j$$

$$= \underline{c_B^T X_B + c_N^T X_N}$$

Basic and Non-Basic Variables

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ w_1 \\ \vdots \\ w_m \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix} = \begin{pmatrix} X_N \\ \\ X_B \end{pmatrix}$$

simplex iterations

N - non-basic

B - basic

$$AX + W = b \iff \sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i$$

$$\boxed{[A \quad I]} \begin{bmatrix} X \\ W \end{bmatrix} = b \iff \sum_{j=1}^{n+m} a_{ij} x_j = b_i$$

new matrix $A^{m \times (n+m)}$

$$\underline{\sum_{j \in B} a_{ij} x_j} + \underline{\sum_{j \in N} a_{ij} x_j} = b_i$$

Basic and Non-Basic Variables

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ w_1 \\ \vdots \\ w_m \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix} = \begin{pmatrix} X_N \\ \\ X_B \end{pmatrix}$$

simplex iterations

N - non-basic

B - basic

$$AX + W = b \iff \sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i$$

$$\boxed{[A \quad I]} \begin{bmatrix} X \\ W \end{bmatrix} = b \iff \underline{\sum_{j \in B} a_{ij} x_j} + \underline{\sum_{j \in N} a_{ij} x_j} = b_i$$

new matrix $A^{m \times (n+m)}$

$$\underline{B X_B} + \underline{N X_N} = b$$

Basic and Non-Basic Variables

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ w_1 \\ \vdots \\ w_m \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix} = \begin{pmatrix} X_N \\ \\ X_B \end{pmatrix}$$

simplex iterations

N - non-basic

B - basic

$$\begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} X \\ W \end{bmatrix} = b \iff \begin{bmatrix} B & N \end{bmatrix} \begin{bmatrix} X_B \\ X_N \end{bmatrix} = b$$

B = basic var. col's of [A I], N = non-basic var. cols of [A I]

Simplex Iterations in Matrix Form

① $Bx_B + Nx_N = b \Longrightarrow \boxed{x_B = B^{-1}b - B^{-1}Nx_N}$

②
$$\begin{aligned}
\zeta &= c_B^T x_B + c_N^T x_N \\
&= c_B^T\left(B^{-1}b - B^{-1}Nx_N\right) + c_N^T x_N \\
&= c_B^T B^{-1}b - \left(c_B^T B^{-1}N - c_N^T\right)x_N \\
&= \boxed{c_B^T B^{-1}b - \left((B^{-1}N)^T c_B - c_N\right)^T x_N} \\
&= \bar{\zeta} + \bar{c}^T x_N
\end{aligned}$$

Simplex Iterations in Matrix Form

$$\boxed{\begin{aligned} \max \quad & \zeta(x) = c^T x \\ \text{s.t.} \quad & w = b - Ax \\ & x, w \geq 0 \end{aligned}}$$

$x \in \mathbb{R}^n$,

$w \in \mathbb{R}^m$

Simplex iterations

$$\boxed{\begin{aligned} \max \quad & \zeta(x) = \underline{c_B^T B^{-1} b} - \underline{\left((B^{-1}N)^T c_B - c_N \right)^T} x_N \\ \text{s.t.} \quad & x_B = \underline{B^{-1} b} - \underline{B^{-1} N} x_N \\ & x_B, x_N \geq 0 \end{aligned}}$$

B, N are basic and non-basic var. cols of $[A \ I]$

Simplex Iterations in Matrix Form

(P)

$c_B^T B^{-1} b$	$-\left((B^{-1}N)^T c_B - c_N\right)^T$
$B^{-1} b$	$-B^{-1} N$

negative transpose

(D)

$-c_B^T B^{-1} b$	$-(B^{-1} b)^T$
$(B^{-1}N)^T c_B - c_N$	$(B^{-1} N)^T$

Simplex Iterations in Matrix Form

(P)

ζ^*	$-(z_N^*)^T$
x_B^*	$-B^{-1}N$

negative transpose

(D)

$-\zeta^*$	$-(x_B^*)^T$
z_N^*	$(B^{-1}N)^T$

Simplex Iterations in Matrix Form

(P)

$$\begin{aligned} \max\ & \zeta(x) = \zeta^* - (z_N^*)^T x_N \\ \text{s.t.}\ & x_B = x_B^* - (B^{-1}N)\, x_N \\ & x_N, x_B \geq 0 \end{aligned}$$

(D)

$$\begin{aligned} \max\ & -\zeta(z) = -\zeta^* - (x_B^*)^T z_B \\ \text{s.t.}\ & z_N = (z_N^*) + (B^{-1}N)^T z_B \\ & z_B, z_N \geq 0 \end{aligned}$$

$\zeta^* = c_B^T B^{-1} b$, $z_N^* = (B^{-1}N)^T c_B - c_N$, $x_B^* = B^{-1} b$

Simplex Iterations in Matrix Form

(P)
$$\begin{aligned} \max \quad & \zeta(x) = \zeta^* - (z_N^*)^T x_N \\ \text{s.t.} \quad & x_B = x_B^* - (B^{-1}N) x_N \\ & x_N, x_B \geq 0 \end{aligned}$$

Optimal Dictionary

$\Updownarrow$

$z_N^* \geq 0$

$(x_B^* \geq 0)$

(D)
$$\begin{aligned} \max \quad & -\zeta(z) = -\zeta^* - (x_B^*)^T z_B \\ \text{s.t.} \quad & z_N = (z_N^*) + (B^{-1}N)^T z_B \\ & z_B, z_N \geq 0 \end{aligned}$$

$\zeta^* = c_B^T B^{-1} b$, $\quad z_N^* = (B^{-1}N)^T c_B - c_N$, $\quad x_B^* = B^{-1} b$

Complementary Variables

$$\begin{array}{c} (w_1 \; - - - - - - \; w_m) \\ (x_1, x_2, \ldots, x_n, \; x_{n+1} \; - - - - \; x_{n+m}) \end{array}$$

$$\updownarrow \qquad\qquad \updownarrow$$

$$\begin{array}{c} (y_1, \; . \; - - - - \; y_m) \\ (z_1, z_2, \ldots, z_n, \; z_{n+1}, \; - - - - \; z_{n+m}) \end{array}$$

① $x_j \sim z_j$ $\qquad$ $w_i \sim y_i$

$(x_j z_j = 0)$ $\qquad$ $(w_i y_i = 0)$

② x_j leaves/enters $\sim$ z_j enters/leaves

w_i leaves/enters $\sim$ y_j enters/leaves

③ $x_N \sim z_N$, $\quad x_B \sim z_B$

[V] p.95

0

$$
\begin{array}{ll}
\text{maximize} & 4x_1 + 3x_2 \\
\text{subject to} & x_1 - x_2 \le 1 \\
& 2x_1 - x_2 \le 3 \\
& x_2 \le 5 \\
& x_1,\ x_2 \ge 0.
\end{array}
$$

$$
\begin{array}{l}
x_1 - x_2 + x_3 = 1 \\
2x_1 - x_2 + x_4 = 3 \\
x_2 + x_5 = 5
\end{array}
$$

The matrix A is given by

$$
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5
\end{array}
$$

$$
\begin{bmatrix}
1 & -1 & 1 & & \\
2 & -1 & & 1 & \\
0 & 1 & & & 1
\end{bmatrix}.
$$

(Note that some zeros have not been shown.) The initial sets of basic and nonbasic indices are

$$
\underline{\mathcal{B} = \{3, 4, 5\}} \quad \text{and} \quad \underline{\mathcal{N} = \{1, 2\}.}
$$

Corresponding to these sets, we have the submatrices of A:

$$
B = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & -1 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}.
$$

Hint Undo

Number format: Fraction Zero Visibility: Visible

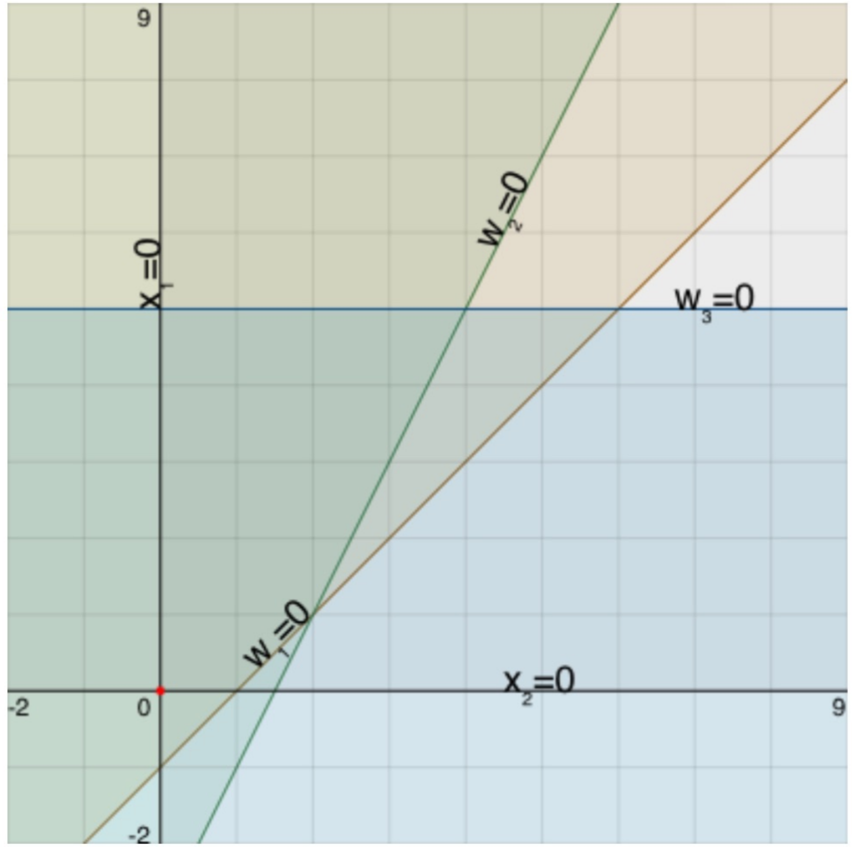
Current Dictionary

maximize	ζ	=			4	x_1	+	3	x_2
subject to:	w_1	=	1	-	1	x_1	-	-1	x_2
	w_2	=	3	-	2	x_1	-	-1	x_2
	w_3	=	5	-	0	x_1	-	1	x_2

$x_1 \quad x_2 \quad w_1 \quad w_2 \quad w_3 \geq 0$

Optimal Infeasible Unbounded

Pick a judge: Bart Simpson



1

x_1 enters, largest coeff.

$$\begin{aligned} \text{maximize} \quad & 4x_1 + 3x_2 \\ \text{subject to} \quad & x_1 - x_2 \le 1 \\ & 2x_1 - x_2 \le 3 \\ & x_2 \le 5 \\ & x_1,\ x_2 \ge 0. \end{aligned}$$

$x_1 \le 1$

$x_1 \le 3/2$

$x_1 \le \infty$

$x_1 - x_2 + x_3 = 1$

$2x_1 - x_2 + x_4 = 3$

$x_2 + x_5 = 5$

The matrix A is given by

X_3 leaves

$$\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & -1 & 1 & & \\ 2 & -1 & & 1 & \\ 0 & 1 & & & 1 \end{bmatrix} \end{array}.$$

(Note that some zeros have not been shown.) The initial sets of basic and nonbasic indices are

$$\mathcal{B} = \{3, 4, 5\} \qquad \text{and} \qquad \mathcal{N} = \{1, 2\}.$$

Corresponding to these sets, we have the submatrices of A:

$$B = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & -1 \\ 2 & -1 \\ 0 & 1 \end{bmatrix}.$$

2

Step 9. The new sets of basic and nonbasic indices are

$$\mathcal{B} = \{1, 4, 5\} \qquad \text{and} \qquad \mathcal{N} = \{3, 2\}.$$

Corresponding to these sets, we have the new basic and nonbasic submatrices of A,

$$B = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 0 & & 1 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & -1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix},$$

and the new basic primal variables and nonbasic dual variables:

$$x^*_{\mathcal{B}} = \begin{bmatrix} x^*_1 \\ x^*_4 \\ x^*_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \qquad z^*_{\mathcal{N}} = \begin{bmatrix} z^*_3 \\ z^*_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}.$$

x₂ enters
x₄ leaves

$$\begin{bmatrix} 1 & -1 & 1 & & \\ 2 & -1 & & 1 & \\ 0 & 1 & & & 1 \end{bmatrix}$$

$$\begin{array}{lllllllllll} \text{maximize} & \zeta & = & 4 & + & -4 & w_1 & + & 7 & x_2 \\ \text{subject to:} & x_1 & = & 1 & - & 1 & w_1 & - & -1 & x_2 \\ & w_2 & = & 1 & - & -2 & w_1 & - & 1 & x_2 \\ & w_3 & = & 5 & - & 0 & w_1 & - & 1 & x_2 \end{array}$$

$$x_1 \quad x_2 \quad w_1 \quad w_2 \quad w_3 \ \geq \ 0$$

3

Step 9. The new sets of basic and nonbasic indices are

$$\mathcal{B} = \{1, 2, 5\} \qquad \text{and} \qquad \mathcal{N} = \{3, 4\}.$$

Corresponding to these sets, we have the new basic and nonbasic submatrices of A,

$$B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and the new basic primal variables and nonbasic dual variables:

$$x^*_{\mathcal{B}} = \begin{bmatrix} x^*_1 \\ x^*_2 \\ x^*_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} \qquad z^*_{\mathcal{N}} = \begin{bmatrix} z^*_3 \\ z^*_4 \end{bmatrix} = \begin{bmatrix} -10 \\ 7 \end{bmatrix}.$$

x_3 enters

x_5 leaves

$$\begin{bmatrix} 1 & -1 & 1 & & \\ 2 & -1 & & 1 & \\ 0 & 1 & & & 1 \end{bmatrix}$$

Current Dictionary

maximize	ζ	=	11	+	10	w_1	+	-7	w_2
subject to:	x_1	=	2	-	-1	w_1	-	1	w_2
	x_2	=	1	-	-2	w_1	-	1	w_2
	w_3	=	4	-	2	w_1	-	-1	w_2

$x_1 \;\; x_2 \;\; w_1 \;\; w_2 \;\; w_3 \geq 0$

Opt.

Step 9. The new sets of basic and nonbasic indices are

$$\underline{\mathcal{B} = \{1, 2, 3\}} \qquad \text{and} \qquad \underline{\mathcal{N} = \{5, 4\}}.$$

Corresponding to these sets, we have the new basic and nonbasic submatrices of A,

$$B = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad N = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},$$

Optimal as

$z_N^* \geq 0$

$(x_B^* \geq 0)$

and the new basic primal variables and nonbasic dual variables:

$$x_{\mathcal{B}}^* = \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 2 \end{bmatrix} \qquad z_{\mathcal{N}}^* = \begin{bmatrix} z_5^* \\ z_4^* \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -1 & 1 & & \\ 2 & -1 & & 1 & \\ 0 & 1 & & & 1 \end{bmatrix}$$

$\zeta^* = 31$

Current Dictionary

maximize $\zeta = 31 + -5$ w $+ -2$ w

subject to:

$x_1 = 4 - 1/2\, w_3 - 1/2\, w_2$

$x_2 = 5 - 1\, w_3 - 0\, w_2$

$w_1 = 2 - 1/2\, w_3 - -1/2\, w_2$

$x_1 \quad x_2 \quad w_1 \quad w_2 \quad w_3 \geq 0$

Hint Undo

Number format: Fraction Zero Visibility: Visible

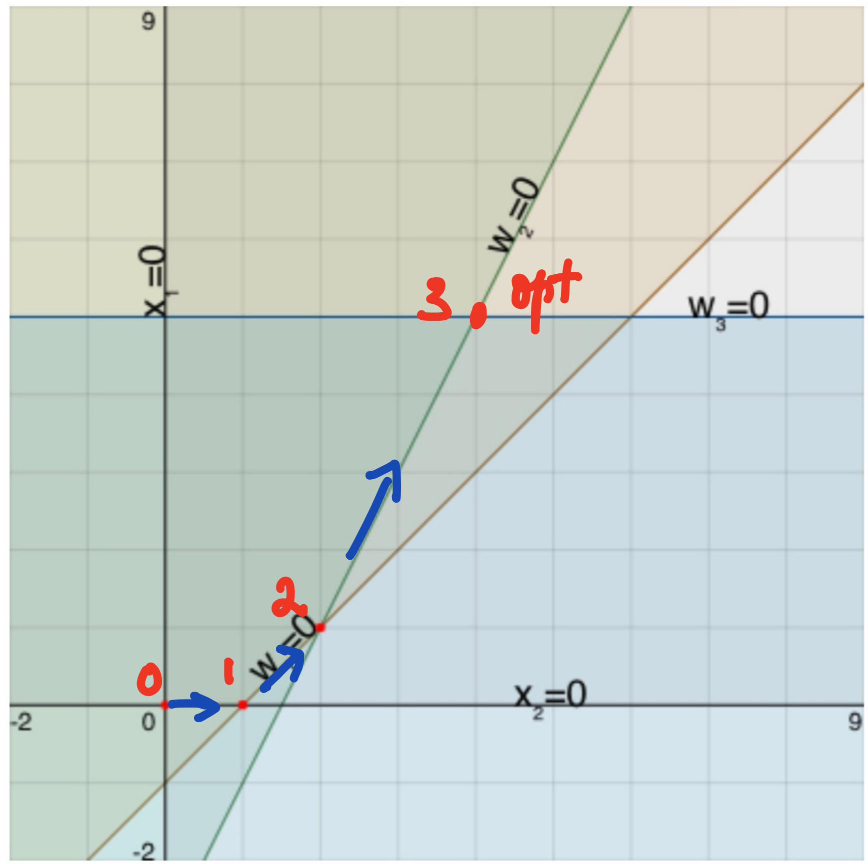
Current Dictionary

maximize	ζ	=	31	+	-5	w	+	-2 w
subject to:	x_1	=	4	-	1/2	w_3	-	1/2 w_2
	x_2	=	5	-	1	w_3	-	0 w_2
	w_1	=	2	-	1/2	w_3	-	-1/2 w_2

$x_1 \quad x_2 \quad w_1 \quad w_2 \quad w_3 \geq 0$

Optional Infeasible Unbounded

Pick a judge: Bart Simpson



$$\zeta^* = 31 \quad \left(= 4x_1^* + 3x_2^* = 4(4) + 3(5) \right)$$