Dual of general LP Problem

([V] p.107 / #6.6 , Chapter 9, p. 152)

Recall (P) $\max \; \zeta = c^T x$

s.t. $Ax \le b$

$0 \le x$

(D) $\min \; \xi = b^T y$

s.t. $A^T y \ge c$

$y \ge 0$

Derivation:

Take linear combination of the constraints.

y — Lagrange Multiplier

$$\left(Ax \le b \Rightarrow y^T(Ax) \le y^T b \Rightarrow (y^T A)\, x \le y^T b \Rightarrow c^T x \le (y^T A)\, x \le y^T b \right)$$

General LP

M1

$$\max \quad c^T x$$
$$\text{s.t.} \quad Ax \le \vec{b}$$
$$x \le \vec{u}$$

(not necessarily $x \ge 0$)

$$\max \quad c^T(x^+ - x^-)$$
$$A(x^+ - x^-) \le \vec{b}$$
$$x^+ - x^- \le \vec{u}$$
$$x^+, x^- \ge 0$$

$x = x^+ - x^-$

$x^+, x^- \ge 0$

$$y^T A(x^+ - x^-) \leq y^T \vec{b} \qquad y \geq 0$$

$$z^T (x^+ - x^-) \leq z^T \vec{u} \qquad z \geq 0$$

$$\underbrace{(y^T A + z^T)}_{\geq c^T} x^+ + \underbrace{(-y^T A - z^T)}_{\geq -c^T} x^- \leq y^T b + z^T \vec{u}$$

(D) $\quad \min \ \xi(y,z) = b^T y + u^T z$

s.t. $\quad A^T y + z \geq c$

$\qquad A^T y + z \leq c \quad \Rightarrow \quad \boxed{A^T y + z = c}$

$\qquad y, z \geq 0$

M2

$$\left\{\begin{array}{ll} \max & \vec{c}^T x \\ \text{s.t.} & A x \le \vec{b} \\ & x \le \vec{u} \end{array}\right.$$

$$\Longleftrightarrow \quad \begin{array}{ll} \max & f(x) = \vec{c}^T x \\ \text{s.t.} & A x - \vec{b} \le 0 \\ & x - u \le 0 \end{array}$$

Lagrangian function

$$L(x, y, z) = \vec{c}^T x - y^T(Ax - b) - z^T(x - u)$$

$$y \ge 0 \qquad z \ge 0$$

Let $\hat{x}$ is (P)-feasible.

$$\zeta(\hat{x}) = c^T\hat{x}$$

$$\le c^T\hat{x} - \underbrace{y^T}_{y \ge 0}\underbrace{(A\hat{x} - b)}_{\le 0} - \underbrace{z^T}_{z \ge 0}\underbrace{(\hat{x} - u)}_{\le 0}$$

$$\le \underbrace{\max_{x}\left(c^T x - y^T(Ax - b) - z^T(x - u)\right)}$$

$$:= \xi(y, z)$$

Weak Duality $\quad \zeta(x) \le \xi(y, z)$

x : P-feasible $\rightarrow \quad \max \zeta(x) \le \min \xi(y, z) \leftarrow \quad y, z \ge 0$

$$\zeta^* \le \xi^*$$

$$(D) \quad \min_{y,z \geq 0} \xi(y,z)$$

$$= \min_{y,z \geq 0} c^T x - y^T(Ax-b) - z^T(x-u)$$

$$= \min_{y,z \geq 0} \underbrace{(c^T - y^T A - z^T)}\, x + y^T b + z^T u$$

$$= \begin{cases} \displaystyle\min_{y,z \geq 0} y^T b + z^T u & \text{if } c^T - y^T A - z^T = 0 \\ +\infty & \text{if } c^T - y^T A - z^T \neq 0 \end{cases}$$

$$
\begin{aligned}
(D) \quad \min \quad & b^T y + u^T z \\
\text{s.t.} \quad & A^T y + z = c \\
& y, z \geq 0
\end{aligned}
$$