[C] p.67

$$\max. \quad \zeta = 40x_1 + 70x_2$$

$$(P) \quad \text{s.t.} \begin{cases} x_1 + x_2 \leq 100 \\ 10x_1 + 30x_2 \leq 4000 \\ x_1, x_2 \geq 0 \end{cases}$$

Opt. solution: $x_1^* = 25$, $x_2^* = 75$, $\zeta^* = 40(25) + 70(75)$
$= 6250$

Q. (1) Check optimality.

(2) What if 4000 is changed to $4000 + t$

How would x_1^*, x_2^*, ζ^* be changed?

How large can t be?

A (1) (find y_1^*, y_2^* and check validity of Strong duality.)

($y_1^* = 32.5$, $y_2^* = 0.75$, given in [C])

(D)

$$\min \quad 100y_1 + 4000y_2$$

$$\text{s.t.} \quad \begin{cases} y_1 + 10y_2 \geq 40 \\ y_1 + 50y_2 \geq 70 \end{cases}$$

$$y_1, y_2 \geq 0$$

(Use complementary slackness) $x_1^* > 0 \Rightarrow z_1^* = 0 \Rightarrow y_1 + 10y_2 = 40$

$x_2^* > 0 \Rightarrow z_2^* = 0 \Rightarrow y_1 + 50y_2 = 70$

$$y_1 = 32.5 \geq 0, \quad y_2 = 0.75 \geq 0$$

(Check:

$$J^* = \underbrace{40(25) + 70(75) = 100(32.5) + \cancel{4000}(0.75)}_{=6250} = \xi^* \text{)}$$

A(2) Use B, N method

$$\zeta = C_B^T B^{-1} b - \left((B^{-1}N)^T C_B - C_N \right)^T X_N$$

$$X_B = B^{-1} b - (B^{-1}N) X_N$$

$$[A \quad I] = \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 10 & 50 & 0 & 1 \end{bmatrix} \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ & & (w_1) & (w_2) \end{array} \end{array}$$

Start

P

		x_1	x_2
	0	40	70
$x_3\ (w_1)$	100	-1	-1
$x_4\ (w_2)$	4000	-10	-50

D

		(z_3) y_1	(z_4) y_2
	0	-100	-4000
z_1	-40	1	10
z_2	-70	1	50

End (Opt)

P

		x_3	x_4
	6250	-32.5	-0.75
x_1	25	$-B^{-1}N$	
x_2	70		

D

		z_1	z_2
	-6250	-25	-70
y_1	32.5	$(B^{-1}N)^T$	
y_2	0.75		

$\mathcal{B} = \{1, 2\}, \quad \mathcal{N} = \{3, 4\}$

$$B = \begin{bmatrix} 1 & 1 \\ 10 & 50 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$B^{-1} = \frac{\begin{pmatrix} 50 & -1 \\ -10 & 1 \end{pmatrix}}{40} = \begin{pmatrix} 5/4 & -1/40 \\ -1/4 & 1/40 \end{pmatrix}$$

End (Opt)

(P)

		x_3	x_4
	6250	-32.5	-0.75
x_1	25	$-5/4$	$1/40$
x_2	70	$1/4$	$-1/40$

(D)

		z_1	z_2
	-6250	-25	-70
y_1	32.5	$5/4$	$-1/4$
y_2	0.75	$-1/40$	$1/40$

$$\zeta = c_B^T B^{-1} b - \left((B^{-1}N)^T c_B - c_N \right)^T x_N$$

$$x_B = B^{-1}b - (B^{-1}N)\, x_N$$

$$\boxed{-3000 \le t \le 1000}$$

Changed to $B^{-1}\left(b + \begin{pmatrix} 0 \\ t \end{pmatrix} \right)$

$$= \begin{pmatrix} 5/4 & -\frac{1}{40} \\ -1/4 & \frac{1}{40} \end{pmatrix} \begin{pmatrix} 100 \\ 4000 + t \end{pmatrix} = \begin{pmatrix} 25 - \frac{t}{40} \\ 75 + \frac{t}{40} \end{pmatrix} \begin{matrix} \ge 0 \\ \ge 0 \end{matrix}$$

$x_1^* \longrightarrow 25 - t/40, \qquad x_2^* \longrightarrow 75 + t/40$

M1 $\quad S^* = 40x_1^* + 70x_2^*$

$$= 40\left(25 - \frac{t}{40}\right) + 70\left(75 + \frac{t}{40}\right)$$

$$= 6250 + \underline{0.75t}$$

M2 $\quad S^* = c_B^T B^{-1}\left(b + \begin{pmatrix} 0 \\ t \end{pmatrix}\right)$

$$= \begin{pmatrix} 40 & 70 \end{pmatrix} \begin{pmatrix} 5/4 & -1/40 \\ -1/4 & 1/40 \end{pmatrix} \begin{pmatrix} 100 \\ 4000+t \end{pmatrix}$$

$$= \underline{\begin{pmatrix} 32.5 & 0.75 \end{pmatrix}} \begin{pmatrix} 100 \\ 4000+t \end{pmatrix}$$

(y_1^*, y_2^*)

no accident.

$$= 6250 + \underline{0.75t}$$

M3

$$
\begin{aligned}
\zeta^*_{new} &= 40x_1^* + 70x_2^* \\
&\le (y_1^* + 10y_2^*)x_1^* + (y_1^* + 50y_2^*)x_2^* \\
&= y_1^*(x_1^* + x_2^*) + y_2^*(10x_1^* + 50x_2^*) \\
&\le y_1^*(100) + y_2^*(4000 + t) \\
&= \underbrace{100y_1^* + 4000y_2^*}_{\text{old}} + \underbrace{y_2^* t}_{\substack{\text{change} \\ = 0.75t}}
\end{aligned}
$$

Note: $\zeta^* = c_B^T B^{-1} b = b^T B^{-T} c_B$

$$\Delta\zeta^* = (\Delta b)^T \underline{(B^{-1} c_B)}$$

$y^* = B^{-1} c_B$

In fact $$\boxed{\zeta^*_{new} = \zeta^*_{old} + (\Delta b)^T y^*_{old}}$$

(if non-degenerate, & $|\Delta b| \ll 1$)

[C] p. 66 Thm 5.5

$$(\Delta b)^T y^*_{old} = (\Delta b)_1 y_1^* + (\Delta b)_2 y_2^* + \cdots + (\Delta b)_m y_m^*$$

(shadow price)

pf (P) $\max \ \zeta = c^T x$

s.t. $Ax \le b$

$x \ge 0$

at Opt:

(P) $\zeta = \zeta^* - z_N^* x_N$

$x_B = x_B^* - (B^{-1}N) x_N$

$$z_N^* = (B^{-1}N)^T c_B - c_N \ge 0$$

$$x_B^* = B^{-1} b \ge 0$$

$$\zeta^* = c_B^T B^{-1} b = c_B^T x_B^* = b^T B^{-T} c_B$$

$$(D) \quad \min \ \xi = b^T y$$
$$\text{s.t.} \quad A^T y \geq c$$
$$y \geq 0$$

at Opt.

$$-\xi = -\zeta^* + (x_B^*)^T z_B$$

$$z_N = z_N^* + (B^{-1}N)^T z_B$$

Compare:

$$\zeta^* = c^T x^* = \xi^* = b^T y^*$$
$$= b^T B^{-T} c_B$$

Guess/Prove $\boxed{y^* = B^{-T} c_B}$

(1) $B^{-T} c_B$ does gives opt. objective value

(2) check for feasibility of $B^{-T} c_B$

$$A^T (B^{-T} c_B) \geq c \quad ?$$

$$B^{-T} c_B \geq 0 \quad ?$$

We need $A^T y \geq \vec{c}$ $\iff$ $y^T A \geq c^T$

$y \geq \vec{0}$ $\iff$ $y^T \geq \vec{0}$

$$y^T [A \quad I] \geq [c^T \quad 0^T]$$

$$y^T [B \quad N] \geq [c_B^T \quad c_N^T]$$

$y^T B \overset{?}{\geq} c_B^T$ $\qquad$ $y^T N \overset{?}{\geq} c_N^T$

$$y = B^{-T} c_B \,, \qquad y^T B = c_B^T B^{-1} B = c_B^T \geq c_B^T$$

$$y^T N = \underbrace{c_B^T B^{-1} N \geq c_N}_{\text{Optimality condition}}$$

$$z_N^* = (B^{-1}N)^T c_B - c_N \geq 0$$