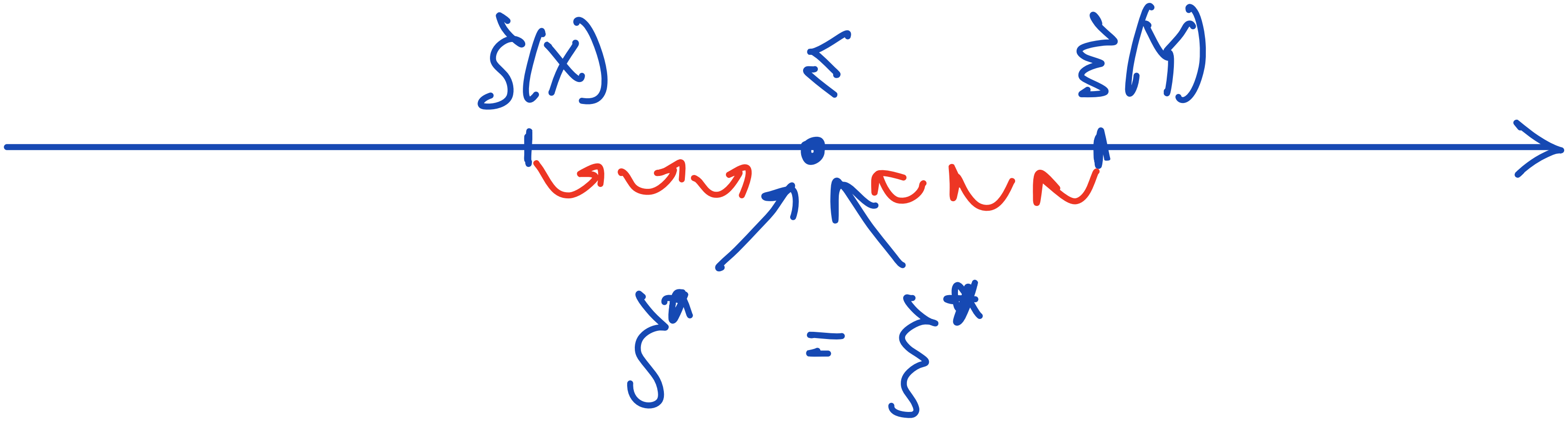
Solution of Primal vs Dual Problems



$$\xi(X) \longrightarrow \xi^* = \xi^* \longleftarrow \xi(Y)$$

Dual Simplex Method [V] p.71

We begin with an example:

Step 0

P

$$
\begin{array}{lrl}
\text{maximize} & -x_1 - x_2 & \\
\text{subject to} & -2x_1 - x_2 & \leq 4 \\
& -2x_1 + 4x_2 & \leq -8 \\
& -x_1 + 3x_2 & \leq -7 \\
& x_1,\ x_2 & \geq 0.
\end{array}
$$

all neg. coeffs.

original not P-feasible

The dual of this problem is

D

$$
\begin{array}{lrl}
\text{minimize} & 4y_1 - 8y_2 - 7y_3 & \\
\text{subject to} & -2y_1 - 2y_2 - y_3 & \geq -1 \\
& -y_1 + 4y_2 + 3y_3 & \geq -1 \\
& y_1,\ y_2,\ y_3 & \geq 0.
\end{array}
$$

Introducing variables w_i, $i = 1, 2, 3$, for the primal slacks and z_j, $j = 1, 2$, for the dual slacks, we can write down the initial primal and dual dictionaries:

Dual Simplex Method [V] p.71

Step 0

(P)

$$
\begin{array}{rcrrrrrrr}
\zeta & = & & - & 1 & x_1 & - & 1 & x_2 \\
\hline
w_1 & = & 4 & + & 2 & x_1 & + & & x_2 \\
w_2 & = & -8 & + & 2 & x_1 & - & 4 & x_2 \\
w_3 & = & -7 & + & & x_1 & - & 3 & x_2
\end{array}
$$

$-(\quad)^T$

(D)

$$
\begin{array}{rcrrrrrrrrr}
-\xi & = & & - & 4 & y_1 & + & 8 & y_2 & + & 7 & y_3 \\
\hline
z_1 & = & 1 & - & 2 & y_1 & - & 2 & y_2 & - & & y_3 \\
z_2 & = & 1 & - & & y_1 & + & 4 & y_2 & + & 3 & y_3
\end{array}
$$

origin is D-feasible

Dual Simplex Method [V] p.71

Step 1

(P)

$$
\begin{array}{rcl}
\zeta &=& -4 - 0.5\,w_2 - 3\,x_2 \\
\hline
w_1 &=& 12 + w_2 + 5\,x_2 \\
x_1 &=& 4 + 0.5\,w_2 + 2\,x_2 \\
w_3 &=& -3 + 0.5\,w_2 - x_2
\end{array}
$$

$-(\quad)^T$

(D)

$$
\begin{array}{rcl}
-\xi &=& 4 - 12\,y_1 - 4\,z_1 + 3\,y_3 \\
\hline
y_2 &=& 0.5 - 1\,y_1 - 0.5\,z_1 - 0.5\,y_3 \\
z_2 &=& 3 - 5\,y_1 - 2\,z_1 + 1\,y_3
\end{array}
$$

Step 2

(P)

$$
\begin{array}{rcl}
\zeta &=& -7 - 1\,w_3 - 4\,x_2 \\
\hline
w_1 &=& 18 + 2\,w_3 + 7\,x_2 \\
x_1 &=& 7 + w_3 + 3\,x_2 \\
w_2 &=& 6 + 2\,w_3 + 2\,x_2
\end{array}
$$

$-(\quad)^T$

Opt. for both →

(D)

$$
\begin{array}{rcl}
-\xi &=& 7 - 18\,y_1 - 7\,z_1 - 6\,y_2 \\
\hline
y_3 &=& 1 - 2\,y_1 - z_1 - 2\,y_2 \\
z_2 &=& 4 - 7\,y_1 - 3\,z_1 - 2\,y_2
\end{array}
$$

Dual Based Phase I Algorithm [V] p.73

$$
\begin{array}{ll}
\text{maximize} & -x_1 + 4x_2 \\
\text{subject to} & -2x_1 - x_2 \le 4 \\
& -2x_1 + 4x_2 \le -8 \\
& -x_1 + 3x_2 \le -7 \\
& x_1,\ x_2 \ge 0.
\end{array}
$$

(P)

$$
\begin{array}{rcl}
\zeta & = & \quad - 1\, x_1 + 4\, x_2 \\
\hline
w_1 & = & 4 + 2\, x_1 + x_2 \\
w_2 & = & -8 + 2\, x_1 - 4\, x_2 \\
w_3 & = & -7 + x_1 - 3\, x_2
\end{array}
$$

(D)

$$
\begin{array}{rcl}
-\xi & = & \quad - 4\, y_1 + 8\, y_2 + 7\, y_3 \\
\hline
z_1 & = & 1 - 2\, y_1 - 2\, y_2 - y_3 \\
z_2 & = & -4 - y_1 + 4\, y_2 + 3\, y_3
\end{array}
$$

origin not feasible for both

Dual Based Phase I Algorithm [V] p.73

$$
\begin{array}{ll}
\text{maximize} & \underline{-x_1 + 4x_2} \\
\text{subject to} & -2x_1 - x_2 \leq 4 \\
& -2x_1 + 4x_2 \leq -8 \\
& {-x_1} + 3x_2 \leq -7 \\
& x_1,\ x_2 \geq 0.
\end{array}
$$

change obj fct to

$-x_1 - x_2$

(P)

$$
\begin{array}{rcl}
\zeta & = & - 1\,x_1 + \overset{-1}{\cancel{4}}\,x_2 \\
\hline
w_1 & = & 4 + 2\,x_1 + \,x_2 \\
w_2 & = & -8 + 2\,x_1 - 4\,x_2 \\
w_3 & = & -7 + \,x_1 - 3\,x_2
\end{array}
$$

(D)

$$
\begin{array}{rcl}
-\xi & = & - 4\,y_1 + 8\,y_2 + 7\,y_3 \\
\hline
z_1 & = & 1 - 2\,y_1 - 2\,y_2 - \,y_3 \\
z_2 & = & \underset{1}{\cancel{-4}} - \,y_1 + 4\,y_2 + 3\,y_3
\end{array}
$$

Now origin is D-feasible

Simplex Method in Matrix Form [V] Ch.6

$$\max \quad \zeta(x) = c^T x$$
$$\text{s.t.} \quad Ax \leq b$$
$$x \geq 0 \qquad (P)$$

$$\max \quad \zeta(x) = \zeta^* - (z_N^*)^T x_N$$
$$x_B = x_B^* - (B^{-1}N)x_N$$

$[A \;\; I] \longrightarrow B$ and N

$$\zeta^* = c_B^T B^{-1} b, \quad z_N^* = (B^{-1}N)^T c_B - c_N, \quad x_B^* = B^{-1} b$$

Simplex Method in Matrix Form [V] Ch.6

$$\max \ -\xi(y) = -b^T y$$

$$\text{s.t.} \quad A^T y \geq c$$

$$y \geq 0$$

(D)

$$\max \ -\xi(z) = -\zeta^* - (x_B^*)^T z_B$$

$$\text{s.t.} \quad z_N = z_N^* + (B^{-1}N)^T z_B$$

$$[A \quad I] \longrightarrow B \text{ and } N$$

$$\zeta^* = c_B^T B^{-1} b, \quad z_N^* = (B^{-1}N)^T c_B - c_N, \quad x_B^* = B^{-1} b$$

Simplex Method in Matrix Form [V] Ch.6

(1) From z_N^* $\Rightarrow$ entering variable x_j

$$\begin{array}{c|c} \zeta^* & -(z_N^*)^T \\ \hline x_B^* & -(B^{-1}N) \end{array}$$

x_j

jth col of $B^{-1}N$

$= (B^{-1}N)e_j$

$:= \Delta x_B$

Simplex Method in Matrix Form [V] Ch.6

(2) Find $\underline{\max t}$ s.t. $\underline{x_B^* - t\,\Delta x_B \geq 0}$

$x_j = 0$

$$
\begin{array}{c|c}
\zeta^* & -\left(z_N^*\right)^T \\
\hline
x_B^* & -\left(B^{-1}N\right)
\end{array}
$$

jth col of $B^{-1}N$

$= \left(B^{-1}N\right)e_j$

$:= \Delta x_B$

determine leaving variable, x_i

Simplex Method in Matrix Form [V] Ch.6

(3) Update $B, N \longrightarrow (B_{old} \setminus \{i\}) \cup \{j\}$

$\qquad\qquad\qquad\quad (N_{old} \setminus \{j\}) \cup \{i\}$

$x_j = 0$

$$
\begin{array}{c|c}
\zeta^* & -(z_N^*)^T \\
\hline
x_B^* & -(B^{-1}N)
\end{array}
$$

jth col of $B^{-1}N$

$= (B^{-1}N)e_j$

$:= \Delta x_B$

determine leaving variable, x_i

Simplex Method in Matrix Form [V] Ch. ~~6~~ 8.3

What is actually done or needed?

Find max t s.t. $x_B^* - t\,\underbrace{\Delta x_B}_{} \geq 0$

?!

$\Delta x_B = B^{-1} N e_j$

$\iff B\,\Delta x_B = N e_j$

Note: $N e_j = j^{th}$ col. of N

$= a_j \leftarrow$ new index, col of B

Simplex Method in Matrix Form [V] Ch. ~~6~~ 8.3

What is actually done or needed?

$$\text{old}\begin{bmatrix} B & N \end{bmatrix} \longrightarrow \text{new}\begin{bmatrix} \tilde{B} & \tilde{N} \end{bmatrix}$$

$$\begin{array}{cc} a_i & a_j \end{array}$$

$$\tilde{B} = B + (\vec{a}_j - \vec{a}_i)\,\vec{e}_i^{\,T}$$

$$= B\left(I + B^{-1}(a_j - a_i)\,e_i^T\right)$$

$$= B\left(I + B^{-1}(\Delta X_B - e_i)\,e_i^T\right)$$

Simplex Method in Matrix Form [V] Ch. ~~6~~ 8.3

$$\tilde{B} = B \underbrace{\left(I + (\Delta x_B - e_i) e_i^T\right)}_{\text{easy to compute its inverse}}$$

already known from current step

easy to compute its inverse

PROPOSITION 8.1. *Given two column vectors u and v for which $1 + v^T u \neq 0$,*

$$(I + uv^T)^{-1} = I - \frac{uv^T}{1 + v^T u}.$$

PROOF. The proof is trivial. We simply multiply the matrix by its supposed inverse and check that we get the identity:

$$\begin{aligned}
(I + uv^T)\left(I - \frac{uv^T}{1 + v^T u}\right) &= I + uv^T - \frac{uv^T}{1 + v^T u} - \frac{uv^T uv^T}{1 + v^T u} \\
&= I + uv^T \left(1 - \frac{1}{1 + v^T u} - \frac{v^T u}{1 + v^T u}\right) \\
&= I,
\end{aligned}$$

Simplex Method in Matrix Form [V] Ch. ~~6~~ 8.3

$$B_k = B_0 E_1 E_2 \cdots E_k$$

$$B_k^{-1} = E_k^{-1} E_{k-1}^{-1} \cdots E_1^{-1} \cancel{B_0^{-1}} \quad I$$