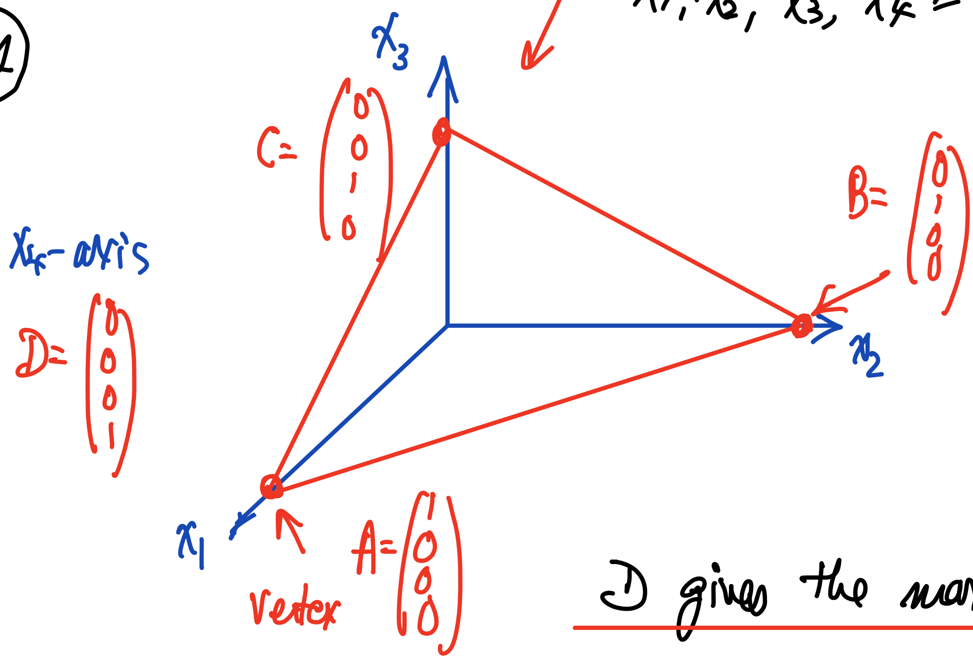
MA 421 Review for Midterm

[V] 2.10

$$\max \quad 6x_1 + 8x_2 + 5x_3 + 9x_4$$
$$\text{s.t.} \quad x_1 + x_2 + x_3 + x_4 = 1$$
$$x_1, x_2, x_3, x_4 \geq 0$$

M1



D gives the maximum value, 9.

M2

$$\max \quad 6x_1 + 8x_2 + 5x_3 + 9x_4$$
$$\text{s.t.} \quad x_1 + x_2 + x_3 + x_4 = 1$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Single out x_4: $\quad x_4 = 1 - x_1 - x_2 - x_3$

Objective fct becomes:

$$\zeta(x_1, x_2, x_3) = 6x_1 + 8x_2 + 5x_3 + 9(1 - x_1 - x_2 - x_3)$$
$$= 9 - 3x_1 - x_2 - 4x_3$$

opt. dictionary

$$\text{s.t.} \quad x_1, x_2, x_3 \geq 0$$

(M3) Consider the dual problem:
(Note the primal problem has only 1 constraint.)

$$\left(\begin{array}{ll} \max & 6x_1 + 8x_2 + 5x_3 + 9x_4 \\ \text{s.t.} & x_1 + x_2 + x_3 + x_4 = 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}\right)$$

$$\min_y \quad 1 \cdot y$$

$$\text{s.t.} \quad \left.\begin{array}{l} y \geq 6 \\ y \geq 8 \\ y \geq 5 \\ y \geq 9 \end{array}\right\} \Rightarrow y \geq 9$$

Hence $y = 9$ is the min.

By complementary slackness $\Rightarrow$

$$\boxed{\begin{array}{ll} x_1 = 0 & (\text{Since } 9 > 6) \\ x_2 = 0 & (\text{Since } 9 > 8) \\ x_3 = 0 & (\text{Since } 9 > 5) \\ x_4 = 1 & (\text{Since } x_1 + x_2 + x_3 + x_4 = 1) \end{array}}$$

[C] 1.2 $\min\ -8x_1 + 9x_2 + 2x_3 - 6x_4 - 5x_5$

s.t. $6x_1 + 6x_2 - 10x_3 + 2x_4 - 8x_5 \geq 3$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

M1 From the coeff. of x_1, x_4

$\Rightarrow$ x_1, x_2 can be as big as possible

$\Rightarrow$ $\min = -\infty$ (unbounded)

M2 Consider dual problem:

($y \geq 0 \Rightarrow 3y \leq 6yx_1 + 6yx_2 - 10yx_3 + 2yx_4 - 8yx_5$
$\leq -8x_1 + 9x_2 + 2x_3 - 6x_4 - 5x_5$)

$$\max_{y \geq 0}\ 3y$$

s.t.
$6y \leq -8$ ← infeasible with $y \geq 0$
$6y \leq 9$
$-10y \leq 2$
$2y \leq -6$ ← infeasible with $y \geq 0$
$-8y \leq -5$

Hence primal problem is unbounded (since primal problem is feasible to start with)

[V] #3.4 $\max \zeta(X) = \sum_{j}^{n} c_j x_j$

M1 s.t. $\sum_j a_{ij} x_j \le 0 \qquad i = 1, \dots, m$

$x_j \ge 0 \qquad j = 1, \dots, n$

Must be feasible (take: $x_1 = x_2 = \dots = x_n = 0$)

Note: $\zeta(0) = 0$

Either $\zeta(0) = 0$ is optimal

Or there is $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ s.t. $\zeta(X) > 0$

If X is feasible, so is tX, $t > 0$.

then $\zeta(tX) = t\zeta(X) \to +\infty$ as $t \to +\infty$

M2 Consider dual problem

$$\min \xi(Y) = 0y_1 + 0y_2 + \dots + 0y_m \ (= 0)$$

s.t. $\sum_i a_{ij} y_i \ge c_j \qquad j = 1, \dots, n$

$y_i \ge 0 \qquad i = 1, \dots, m$

Note that the dual objective function is always 0

Hence

if the dual problem is feasible, then $\xi^* = 0$

$$\Longrightarrow \xi^* = 0$$

if the dual problem is not feasible, then $\xi^* = +\infty$
(unbounded)

(Since the primal problem is always feasible.)

[C]

2.2 Use the simplex method to describe *all* the optimal solutions of the following problem:

$$
\begin{array}{ll}
\text{maximize} & 2x_1 + 3x_2 + 5x_3 + 4x_4 \\
\text{subject to} & x_1 + 2x_2 + 3x_3 + x_4 \le 5 \quad \leftarrow w_1 \\
& x_1 + x_2 + 2x_3 + 3x_4 \le 3 \quad \leftarrow w_2 \\
& x_1, x_2, x_3, x_4 \ge 0.
\end{array}
$$

Use simplex $\Longrightarrow$ optimal dictionary:

$$
\begin{aligned}
\zeta &= 8 - w_1 - w_2 \\
x_2 &= 1 + x_1 - 2w_1 + 3w_2 + 7x_4 \\
x_3 &= 1 - x_1 + w_1 - 2w_2 - 5x_4
\end{aligned}
$$

Solution: $\zeta^* = 8$, $w_1 = 0$, $w_2 = 0$

$$
\begin{aligned}
x_2 &= 1 + x_1 + 7x_4 \\
x_3 &= 1 - x_1 - 5x_4
\end{aligned}
$$

All solutions:

$$
\boxed{
\begin{aligned}
& x_1, x_2, x_3, x_4 \ge 0 \\
& x_2 = 1 + x_1 + 7x_4 \\
& x_3 = 1 - x_1 - 5x_4
\end{aligned}
}
$$

[v] #5.1

$$
\begin{aligned}
\max \quad & x_1 - 2x_2 \\
\text{s.t.} \quad & -x_1 - 2x_2 + x_3 - x_4 \leq 0 \\
& 4x_1 + 3x_2 + 4x_3 - 2x_4 \leq 3 \\
& -x_1 - x_2 + 2x_3 + x_4 = 1 \\
& x_2, x_3 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
-y_1 x_1 - 2y_1 x_2 + y_1 x_3 - y_1 x_4 &\leq 0 \qquad y_1 \geq 0 \\
4y_2 x_1 + 3y_2 x_2 + 4y_2 x_3 - 2y_2 x_4 &\leq 3y_2 \qquad y_2 \geq 0 \\
-y_3 x_1 - y_3 x_2 + 2y_3 x_3 + y_3 x_4 &= y_3
\end{aligned}
$$

$$
\underbrace{(-y_1+4y_2-y_3)x_1 + (-2y_1+3y_2-y_3)x_2 + (y_1+4y_2+2y_3)x_3 + (-y_1-2y_2+y_3)x_4}_{\geq\ x_1 - 2x_2} \leq 3y_2 + y_3
$$

Hence

$$
\begin{aligned}
x_1 &\leq (-y_1+4y_2-y_3)x_1 \Rightarrow -y_1+4y_2-y_3 = 1 \\
-2x_2 &\leq (-2y_1+3y_2-y_3)x_2 \overset{(x_2 \geq 0)}{\Longrightarrow} -2y_1+3y_2-y_3 \geq -2 \\
0 &\leq (y_1+4y_2+2y_3)x_3 \overset{(x_3 \geq 0)}{\Longrightarrow} y_1+4y_2+2y_3 \geq 0 \\
0 &\leq (-y_1-2y_2+y_3)x_4 \Rightarrow -y_1-2y_2+y_3 = 0
\end{aligned}
$$

Dual Problem:

$$\min \quad 3y_2 + y_3$$

s.t.

$$-y_1 + 4y_2 - y_3 = 1$$

$$-2y_1 + 3y_2 - y_3 \geq -2$$

$$y_1 + 4y_2 + 2y_3 \geq 0$$

$$-y_1 - 2y_2 + y_3 = 0$$

$$y_1, y_2 \geq 0$$

5.6 Solve the following linear program:

negative coeffs.

$$
\begin{array}{llrcr}
\text{maximize} & -x_1 - 2x_2 & & & \\
\text{subject to} & -2x_1 + 7x_2 & \leq & 6 \\
& -3x_1 + x_2 & \leq & -1 \\
& 9x_1 - 4x_2 & \leq & 6 \\
& x_1 - x_2 & \leq & 1 \\
& 7x_1 - 3x_2 & \leq & 6 \\
& -5x_1 + 2x_2 & \leq & -3 \\
& x_1,\ x_2 & \geq & 0.
\end{array}
$$

Origin not feasible.

Consider dual problem:

$$
\begin{array}{ll}
\min & 6y_1 - y_2 + 6y_3 + y_4 + 6y_5 - 3y_6 \\
\text{s.t.} & -2y_1 - 3y_2 + 9y_3 + y_4 + 7y_5 - 5y_6 \geq -1 \\
& 7y_1 + y_2 - 4y_3 - y_4 - 3y_5 + 2y_6 \geq -2 \\
& y_1, \cdots, y_6 \geq 0
\end{array}
$$

Now the origin is feasible for dual problem.

[v] #5.14

(P₁) $\max_{x} \; c^T x$

s.t. $Ax \le b$ ← y^T

$0 \le x \le u$ ← z^T

Opt soln: x^*

(D₁) $\min_{y,z} \; y^T b + z^T u$

s.t. $y^T A + z^T \ge c^T$

$y, z \ge 0$

Opt. soln: $y^* = y^*(b)$, z^*

(P₂) $\max_{x,b} \; c^T x - p^T b$

s.t. $Ax - b \le 0$ ← y^T

$0 \le x \le u$ ← z^T

Opt. Soln: x^*, b^*

(D₂) $\min_{y,z} \; z^T u$

s.t. $y^T A + z^T \ge c^T$

$y^T \le p^T$

Opt. Soln: y^*, z^*

Prove: $y^*(b^*) = p$

b^* from (D₂)

Solve (D_2): $\min z^T u \implies$ make z^T as small as possible

From $y^T A + z^T \geq c^T$

$$\implies z^T \geq c^T - y^T A \geq c^T - p^T A$$

Smallest z can be.

make y^T as big as possible

$y^T \leq \underline{\underline{p^T}}$

$$\boxed{\text{Assume } c^T - p^T A > 0}$$

$\implies$ Set $z_*^T = c^T - p^T A > 0$

$y_*^T = p^T > 0$

Solution of (D_2)

By Complementary Slackness:

$z_*^T > 0 \implies \underline{X_* = U}$

$y_*^T > 0 \implies A X_* - b_* = 0$

i.e. $\underline{b_* = AU}$

Solution of (P_2)

Now Solve (D_1) with $b = b^* = Au$, from (D_2)

$$\min \; y^T b + z^T u$$

$$\text{s.t.} \; y^T A + z^T \geq c^T$$

obj fct $= y^T(Au) + z^T u$

$$= (y^T A + z^T)\, u$$

$$\geq \underline{c^T u}$$

Let $\underline{y^T = p^T, \quad z^T = c^T - p^T A}$

Then $y^T b^* + z^T u$

$$= p^T A u + (c^T - p^T A)\, u$$

$$= \underline{c^T u}$$

Hence $c^T u$ is achieved by

$y^T = p^T$ and $z^T = c^T - p^T A$ ← Optimal Solution

$$\boxed{y^* = y^*(b^*) = p}$$

Justification of $c^T - p^T A > 0$

$$\underbrace{\sum_i p_i a_{ij}}_{\text{cost to make 1 unit of } j^{th} \text{ product}} < \underbrace{c_j}_{\text{profit of 1 unit of } j^{th} \text{ product}}$$

p_i: price/cost of 1 unit of i^{th} raw material

a_{ij}: a_{ij} of i^{th} material to make 1 unit of j^{th} product

c_j: profit of j^{th} product (1 unit)

$$\boxed{\text{cost} < \text{profit}}$$

6.1 Consider the following linear programming problem:

$$
\begin{array}{rrcl}
\text{maximize} & -6x_1 + 32x_2 - 9x_3 & & \\
\text{subject to} & -2x_1 + 10x_2 - 3x_3 & \leq & -6 \\
& x_1 - 7x_2 + 2x_3 & \leq & 4 \\
& x_1,\ x_2,\ x_3 & \geq & 0.
\end{array}
$$

Suppose that, in solving this problem, you have arrived at the following dictionary:

$$
\begin{array}{rcrcrcrcr}
\zeta & = & -18 & - & 3\,x_4 & + & 2\,x_2 & & \\
\hline
x_3 & = & 2 & - & x_4 & + & 4\,x_2 & - & 2\,x_5 \\
x_1 & = & 0 & + & 2\,x_4 & - & x_2 & + & 3\,x_5
\end{array}
$$

(a) Which variables are basic? Which are nonbasic?
(b) Write down the vector, $x^*_{\mathcal{B}}$, of current primal basic solution values.
(c) Write down the vector, $z^*_{\mathcal{N}}$, of current dual nonbasic solution values.
(d) Write down $B^{-1}N$.
(e) Is the primal solution associated with this dictionary feasible?
(f) Is it optimal?
(g) Is it degenerate?

(a) Basic vars: X_3, X_1; Non-basic vars: X_4, X_2, X_5

(b) $X^*_B = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ (c) $Z^*_N = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$

(d) $B^{-1}N = \begin{pmatrix} 1 & -4 & 2 \\ -2 & 1 & -3 \end{pmatrix}$

(e) Yes as $X^*_B \geqslant 0$

(f) No as $Z^*_N \not\geqslant 0$

(g) Yes as 2^{nd} entry of $X^*_B = 0$

6.6 Find the dual of the following linear program:

$$\begin{array}{lrcccl} \text{maximize} & & & c^T x & & \\ \text{subject to} & a & \leq & Ax & \leq & b \\ & l & \leq & x & \leq & u. \end{array}$$

$p^T (Ax \leq b)$ $\quad p \geq 0$

$q^T (-Ax \leq -a)$ $\quad q \geq 0$

$r^T (x \leq u)$ $\quad r \geq 0$

$s^T (-x \leq -l)$ $\quad s \geq 0$

$$c^T x \leq (p^T A - q^T A + r^T - s^T) x \leq p^T b - q^T a + r^T u - s^T l$$

$$\boxed{\begin{array}{ll} \min & p^T b - q^T a + r^T u - s^T l \\ \text{s.t.} & p^T A - q^T A + r^T - s^T = c^T \\ & p, q, r, s \geq 0 \end{array}}$$

Same as [V] p.152 (9.2)

7.1 The final dictionary for

$$\begin{array}{lrcrcrcrcl} \text{maximize} & x_1 & + & 2x_2 & + & x_3 & + & x_4 & & \\ \text{subject to} & 2x_1 & + & x_2 & + & 5x_3 & + & x_4 & \le & 8 \\ & 2x_1 & + & 2x_2 & & & + & 4x_4 & \le & 12 \\ & 3x_1 & + & x_2 & + & 2x_3 & & & \le & 18 \\ & & & & x_1, & x_2, & x_3, & x_4 & \ge & 0 \end{array}$$

is

$$\begin{array}{rcl} \zeta & = & 12.4 - \boxed{1.2\, x_1 - 0.2\, x_5 - 0.9\, x_6 - 2.8\, x_4} \\ \hline x_2 & = & 6 - x_1 - 0.5\, x_6 - 2\, x_4 \\ x_3 & = & 0.4 - 0.2\, x_1 - 0.2\, x_5 + 0.1\, x_6 + 0.2\, x_4 \\ x_7 & = & 11.2 - 1.6\, x_1 + 0.4\, x_5 + 0.3\, x_6 + 1.6\, x_4 \end{array}$$

(the last three variables are the slack variables).

(a) What will be an optimal solution to the problem if the objective function is changed to

$$3x_1 + 2x_2 + x_3 + x_4?$$

$$(c_1\ c_2\ c_3\ c_4\ c_5\ c_6\ c_7) = (1\ 2\ 1\ 1\ 0\ 0\ 0)$$

$$\mathcal{B} = \{2, 3, 7\}, \quad \mathcal{N} = \{1, 5, 6, 4\}$$

$$c_{\mathcal{B}} = (2, 1, 0)^T, \quad c_{\mathcal{N}} = (1, 0, 0, 0)^T$$

$$x_{\mathcal{B}}^* = B^{-1}b = \begin{pmatrix} 6 \\ 0.4 \\ 11.2 \end{pmatrix}$$

$$z_{\mathcal{N}}^* = (B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}} = (1.2, 0.2, 0.9, 2.8)^T$$

$$B^{-1}N = \begin{pmatrix} 1 & 0 & 0.5 & 2 \\ 0.2 & 0.2 & -0.1 & -0.2 \\ 1.6 & -0.4 & -0.3 & -1.6 \end{pmatrix}$$

C_1 changes from 1 to 3

$$\Delta C_1 = 2 \Rightarrow \Delta C_N = (2, 0, 0, 0)^T$$

C_B no change, $(\Delta C_B = 0)$

$$\Delta Z_N^* = -\Delta C_N = (-2, 0, 0, 0)^T$$

$$Z_N^*(\text{new}) = Z_N^*(\text{old}) + \Delta Z_N^*$$

$$= (1.2,\ 0.2,\ 0.9,\ 2.8)^T + (-2,\ 0,\ 0,\ 0)^T$$

$$= (-0.8,\ 0.2,\ 0.9,\ 2.8)^T \not\geq 0$$

Hence dictionary is not opt. for the new C.

But we can still start from here:

$$\begin{array}{rcl}
\zeta & = & 12.4 \cancel{-} \overset{+0.8}{\cancel{1.2}}\, x_1 - 0.2\, x_5 - 0.9\, x_6 - 2.8\, x_4 \\
\hline
x_2 & = & 6 - x_1 - 0.5\, x_6 - 2\, x_4 \\
x_3 & = & 0.4 - 0.2\, x_1 - 0.2\, x_5 + 0.1\, x_6 + 0.2\, x_4 \\
x_7 & = & 11.2 - 1.6\, x_1 + 0.4\, x_5 + 0.3\, x_6 + 1.6\, x_4
\end{array}$$

X_1 enters and X_3 leaves (use Simplex tool to continue)

(b) What will be an optimal solution to the problem if the objective function is changed to

$$x_1 + 2x_2 + 0.5x_3 + x_4?$$

C_3 changes from 1 to 0.5

$$\Delta C_3 = -0.5 \implies \Delta C_B = (0, -0.5, 0)^T$$

$$Z_N^* = (B^{-1}N)^T C_B - C_N = (1.2,\ 0.2,\ 0.9,\ 2.8)^T$$

$C_B = (C_2, C_3, C_7)^T$

C_N: no change

$$\Delta Z_N^* = (B^{-1}N)^T \Delta C_B$$

$$= \begin{bmatrix} 1 & 0 & 0.5 & 2 \\ 0.2 & 0.2 & -0.1 & -0.2 \\ 1.6 & -0.4 & -0.3 & -1.6 \end{bmatrix}^T \begin{bmatrix} 0 \\ -0.5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -0.1 & -0.1 & 0.05 & 0.1 \end{bmatrix}^T$$

Hence $Z_N^*(\text{new}) = Z_N^*(\text{old}) + \Delta Z_N^*$

$$= (1.2\ \ 0.2\ \ 0.9\ \ 2.8)^T + (-0.1\ \ -0.1\ \ 0.05\ \ 0.1)^T$$

$$= (1.1\ \ 0.1\ \ 0.95\ \ 2.9)^T > 0$$

still opt.

$$\zeta^* = C_B^T B^{-1} b$$

$$\zeta^*(\text{new}) = \zeta^*(\text{old}) + \Delta C_B^T B^{-1} b$$

$$= 12.4 + (0 \quad -0.5 \quad 0)\begin{pmatrix} 6 \\ 0.4 \\ 11.2 \end{pmatrix}$$

$$= 12.4 - 0.2$$

$$= 12.2$$

Still, $(x_1, x_2, x_3, x_4) = (0, 6, 0.4, 0)$

(c) What will be an optimal solution to the problem if the second constraint's right-hand side is changed to 26?

b_2 changes from 12 to 26

The only thing needs to be checked is x_B^*

$z_N^* = (B^{-1}N)^T C_B - C_N$ is not changed

$$x_B^* = B^{-1} b, \qquad \Delta x_B^* = B^{-1} \Delta b$$

$\mathcal{B} = (2, 3, 7)$

$$B = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}, \quad B^{-1}\Delta b = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 14 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ -1.4 \\ -4.2 \end{bmatrix}$$

$$X_B^*(\text{new}) = X_B^*(\text{old}) + \Delta X_B^*$$

$$= \begin{pmatrix} 6 \\ 0.4 \\ 11.2 \end{pmatrix} + \begin{pmatrix} 7 \\ -1.4 \\ -4.2 \end{pmatrix} = \begin{pmatrix} 13 \\ -1 \\ 7 \end{pmatrix} \cancel{\geq} 0$$

not opt.

Start from:

$$\begin{array}{rcl}
\zeta & = & 12.4 - 1.2\,x_1 - 0.2\,x_5 - 0.9\,x_6 - 2.8\,x_4 \\
\hline
x_2 & = & \cancel{6}\ 13 - x_1 - 0.5\,x_6 - 2\,x_4 \\
x_3 & = & \cancel{0.4}\ -1 - 0.2\,x_1 - 0.2\,x_5 + 0.1\,x_6 + 0.2\,x_4 \\
x_7 & = & \cancel{11.2}\ 7 - 1.6\,x_1 + 0.4\,x_5 + 0.3\,x_6 + 1.6\,x_4
\end{array}$$

dictionary not feasible anymore but dual feasible

So start from its dual version:

$$\begin{array}{c|cccc}
-\xi & -12.4 & -13 & 1 & -7 \\
\hline
 & 1.2 & 1 & 0.2 & 1.6 \\
 & 0.2 & 0 & 0.2 & -0.4 \\
 & 0.9 & 0.5 & -0.1 & -0.3 \\
 & 2.8 & 2 & -0.2 & -1.6
\end{array}$$

use simplex to continue ....

7.2 For each of the objective coefficients in the problem in Exercise 7.1, find the range of values for which the final dictionary will remain optimal.

The only thing needs to be checked is to make sure $Z_N^*(\text{new}) = Z_N^*(\text{old}) + \Delta Z_N^* > 0$

$$Z_N^*(\text{old}) = (B^{-1}N)^T C_B - C_N = (1.2,\ 0.2,\ 0.9\ \ 2.8)^T$$

$$\Delta Z_N^* = (B^{-1}N)^T \Delta C_B - \Delta C_N$$

$$B = \{2, 3, 7\}, \qquad N = \{1, 5, 6, 4\}$$

ΔC_1 : $(1.2\ \ 0.2\ \ 0.9\ \ 2.8)^T - (\Delta C_1\ \ 0\ \ 0\ \ 0)^T$

$$= (1.2 - \Delta C_1\ \ 0.2\ \ 0.9\ \ 2.8)^T > 0$$

$$\Longrightarrow \boxed{\Delta C_1 < 1.2}$$

ΔC_2 : $(1.2\ \ 0.2\ \ 0.9\ \ 2.8)^T + (B^{-1}N)^T \begin{pmatrix} \Delta C_2 \\ 0 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} 1.2 \\ 0.2 \\ 0.9 \\ 2.8 \end{pmatrix} + \begin{bmatrix} 1 & 0 & 0.5 & 2 \\ 0.2 & 0.2 & -0.1 & -0.2 \\ 1.6 & -0.4 & -0.3 & -1.6 \end{bmatrix}^T \begin{pmatrix} \Delta C_2 \\ 0 \\ 0 \end{pmatrix}$$

$$= (1.2\ \ 0.2\ \ 0.9\ \ 0.8)^T + (1\ \ 0\ \ 0.5\ \ 2)^T \Delta C_2$$

$$= \begin{pmatrix} \underline{1.2 + \Delta C_2} & 0.2 & \underline{0.9 + 0.5\Delta C_2} & \underline{2.8 + 2\Delta C_2} \end{pmatrix} \geq 0$$

$$\Longrightarrow \boxed{\Delta C_2 > -1.2}$$

$$\Delta C_2 > -1.8$$

$$\Delta C_2 > -1.4$$

$\underline{\Delta C_3:}$ $$(1.2 \;\; 0.2 \;\; 0.9 \;\; 2.8)^T + (B^{-1}N)^T \begin{pmatrix} 0 \\ \Delta C_3 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1.2 \\ 0.2 \\ 0.9 \\ 2.8 \end{pmatrix} + \begin{bmatrix} 1 & 0 & 0.5 & 2 \\ 0.2 & 0.2 & -0.1 & -0.2 \\ 1.6 & -0.4 & -0.3 & -1.6 \end{bmatrix}^T \begin{pmatrix} 0 \\ \Delta C_3 \\ 0 \end{pmatrix}$$

$$= (1.2 \;\; 0.2 \;\; 0.9 \;\; 2.8)^T + (0.2 \;\; 0.2 \;\; -0.1 \;\; -0.2)^T \Delta C_3$$

$$= \begin{pmatrix} 1.2 + 0.2\Delta C_3 & 0.2 + 0.2\Delta C_3 & 0.9 - 0.1\Delta C_3 & 2.8 - 0.2\Delta C_3 \end{pmatrix} \geq 0$$

$$\left.\begin{aligned} \Delta C_3 &> -6 \\ \Delta C_3 &> -1 \\ \Delta C_3 &< 9 \\ \Delta C_3 &< 14 \end{aligned}\right\} \quad \boxed{-1 < \Delta C_3 < 9}$$

$\underline{\Delta C_4:}$ $$(1.2 \;\; 0.2 \;\; 0.9 \;\; 2.8)^T - (0 \;\; 0 \;\; 0 \;\; \Delta C_4)^T \geq 0$$

$$\boxed{\Delta C_4 < 2.8}$$

6.7 (a) Let A be a given $m \times n$ matrix, c a given n-vector, and b a given m-vector. Consider the following max–min problem:

$$\max_{x \geq 0} \min_{y \geq 0} \left(c^T x - y^T A x + b^T y \right).$$

real number, changed to $y^T b$

By noting that the inner optimization can be carried out explicitly, show that this problem can be reduced to a linear programming problem. Write it explicitly.

(b) What linear programming problem do you get if the min and max are interchanged?

(a) $$\max_{x \geq 0} \underbrace{\min_{y \geq 0} c^T x + y^T(-Ax + b)}$$

$$= c^T x + \underbrace{\min_{y \geq 0} y^T(-Ax+b)}$$

$$= \begin{cases} 0 & \text{if } -Ax+b \geq 0 \\ -\infty & \text{if } \underline{-Ax+b<0} \end{cases}$$

all entries ≥ 0

$$= \begin{cases} \underline{c^T x \quad \text{if } -Ax+b \geq 0} \\ -\infty \quad \text{if } \underline{-Ax+b<0} \end{cases}$$

$$\boxed{\max_{x \geq 0} \; c^T x \qquad Ax \leq b}$$

to be exact, just need any one entry to be negative

(b) $$\min_{y \geq 0} \max_{x \geq 0} c^T x + y^T(-Ax + b)$$

$$= \min_{y \geq 0} \max_{x \geq 0} y^T b + (c^T - y^T A) x$$

$$= \min_{y \geq 0} \left[y^T b + \max_{x \geq 0} (c^T - y^T A) x \right]$$

$$= \min_{y \geq 0} \left[y^T b + \begin{cases} +\infty & \text{if } c^T - y^T A > 0 \\ 0 & \text{if } c^T - y^T A \leq 0 \end{cases} \right]$$

just need anyone entry > 0

all entries ≤ 0

$$\boxed{\begin{aligned} &\min_{y \geq 0} \; y^T b \\ &\text{s.t.} \;\; y^T A \geq c^T \end{aligned}}$$

[C] #9.2

What is the dual of: maximize $\mathbf{cx}$ subject to $\mathbf{Ax} \le \mathbf{b}$? The dual of: maximize $\mathbf{cx}$ subject to $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \ge 0$?

(i) $\max \quad c^T x$

s.t. $\quad Ax \le b \quad \xleftarrow{} y^T$ $\qquad y \ge 0$

$$c^T x \le (y^T A) x \le y^T b$$

$$\boxed{\text{Dual Problem:} \quad \begin{array}{ll} \min & y^T b \\ \text{s.t.} & y^T A \ge c^T \\ & y \ge 0 \end{array}}$$

(ii) $\max \quad c^T x$

s.t. $\quad Ax = b \quad \xleftarrow{} y^T$

$\quad x \ge 0$

$$c^T x \le y^T A x = y^T b$$

$$\boxed{\text{Dual Problem:} \quad \begin{array}{ll} \min & y^T b \\ \text{s.t.} & y^T A \ge c^T \end{array}}$$

Extra Thought:

$$(P) \quad \max \ c^T X \quad \text{s.t.} \quad AX \le b \ \ (y \ge 0), \quad X \ge 0$$

$$\boxed{(D) \quad \min \ y^T b \quad \text{s.t.} \quad y^T A \ge c^T, \quad y \ge 0}$$

$$c^T X \le y^T A X \le y^T b$$

$$(P') \quad \max \ c^T X \quad \text{s.t.} \quad AX \le b \ \ (y \ge 0), \quad -X \le 0 \ \ (z \ge 0)$$

$$c^T X \le (y^T A - z^T) X \le y^T b$$

$$(D') \quad \min \ y^T b \quad \text{s.t.} \quad y^T A - z^T = c^T, \quad y, z \ge 0$$

$$y^T A = c^T + z^T \ge c^T$$

$$\boxed{(D') \quad \min \ y^T b \quad \text{s.t.} \quad y^T A \ge c^T, \quad y \ge 0} \Longleftrightarrow (D)$$

9.5 Generalize Theorems 5.2 and 5.3 to the context of general LP problems. Use your result to find out if

$$x_1^* = 3, \quad x_2^* = -1, \quad x_3^* = 0, \quad x_4^* = 2$$

is an optimal solution of the problem

$$\begin{array}{ll} \text{maximize} & 6x_1 + x_2 - x_3 - x_4 \\ \text{subject to} & x_1 + 2x_2 + x_3 + x_4 \le 5 \\ & 3x_1 + x_2 - x_3 \le 8 \\ & x_2 + x_3 + x_4 = 1 \\ & x_3, x_4 \ge 0. \end{array}$$

$6x_1 + x_2 - x_3 - x_4$ $\quad$ $x_3, x_4 \geq 0$

(*) $\leq (y_1+3y_2)x_1 + (2y_1+y_2+y_3)x_2 + (y_1-y_2+y_3)x_3 + (y_1 \quad +y_3)x_4$

$\leq 5y_1 + 8y_2 + y_3$ $\quad$ $y_1 \geq 0, y_2 \geq 0.$

$$\boxed{\begin{array}{lll} (D) & \min & 5y_1 + 8y_2 + y_3 \\ & \text{s.t.} & y_1 + 3y_2 = 6 \\ & & 2y_1 + y_2 + y_3 = 1 \\ (**) \rightarrow & & y_1 - y_2 + y_3 \geq -1 \\ & & y_1 \quad + y_3 \geq -1 \\ & & y_1, y_2 \geq 0 \end{array}}$$

(x_1 has no sign)
(x_2 has no sign)
($x_3 \geq 0$)
($x_4 \geq 0$)

$x_1^* = 3^{\neq 0}, \quad x_2^* = -1^{\neq 0}, \quad x_3^* = 0, \quad x_4^* = 2^{>0}$

$\boxed{\text{To find } y^*, \text{ we need tightness of } (*):}$

$$6x_1 + x_2 - x_3 - x_4 \qquad x_3, x_4 \geq 0$$

$$(*) \leq (y_1 + 3y_2)x_1 + (2y_1 + y_2 + y_3)x_2 + (y_1 - y_2 + y_3)x_3 + (y_1 + y_3)x_4$$

$$\leq 5y_1 + 8y_2 + y_3 \qquad y_1 \geq 0, \; y_2 \geq 0.$$

i.e.

$$\begin{cases} y_1 + 3y_2 = 6 \\ 2y_1 + y_2 + y_3 = 1 \\ y_1 + y_3 = -1 \end{cases} \Rightarrow \begin{array}{l} y_1 + 3y_2 = 6 \\ y_1 + y_2 = 2 \end{array}$$

$$\Rightarrow \begin{array}{l} y_2 = 2 \\ y_1 = 0 \\ y_3 = -1 \end{array}$$

check (**) $\quad y_1 - y_2 + y_3 = 0 - 2 - 1 = -3 \cancel{\geq} -1$

$\boxed{\text{Hence } x^* \text{ is not optimal}}$