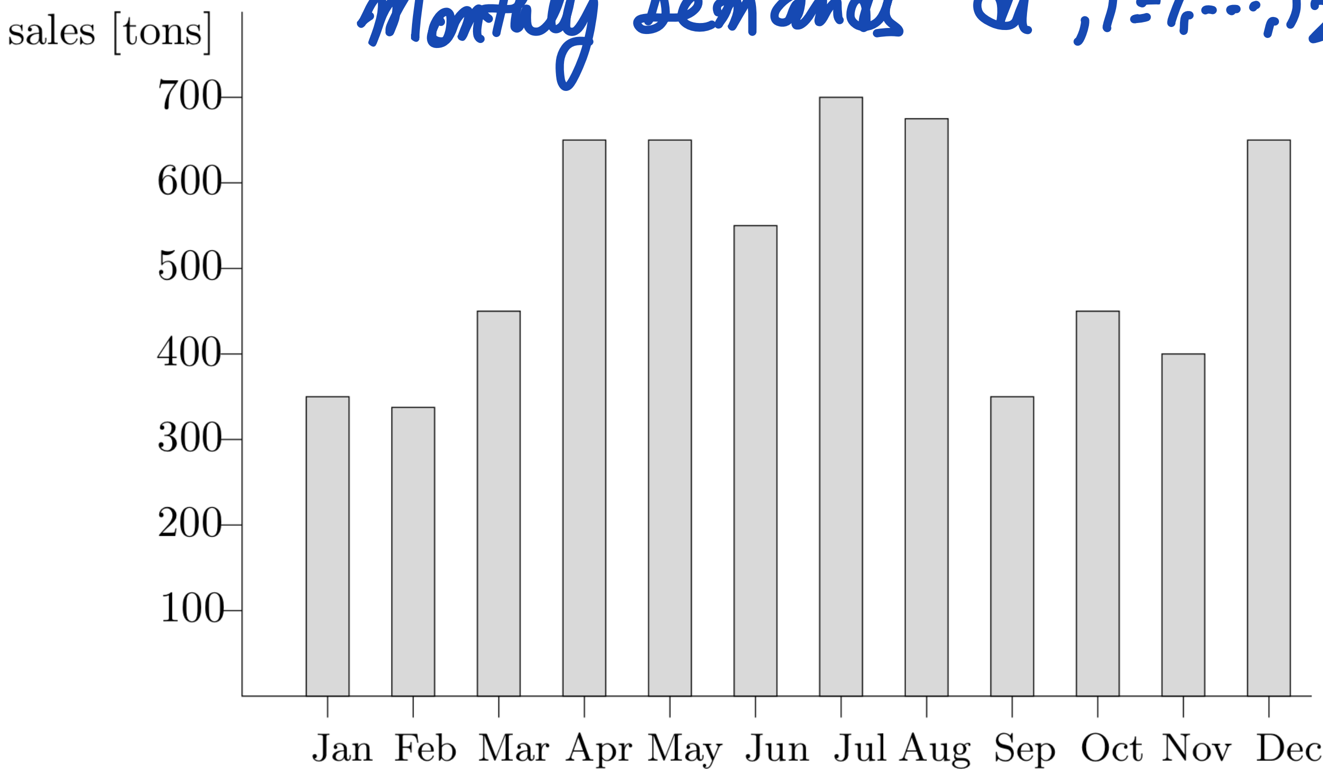
Production Planning

[MG] p.16

2.3 Ice Cream All Year Round

The next application of linear programming again concerns food (whi should not be surprising, given the importance of food in life and the dif culties in optimizing sleep or love). The ice cream manufacturer Icicle Wor Ltd.[2] needs to set up a production plan for the next year. Based on histor extensive surveys, and bird observations, the marketing department has con up with the following prediction of monthly sales of ice cream in the ne year:

Monthly Demands d_i, $i=1,\dots,12$



Production Planning

[MG] p.16

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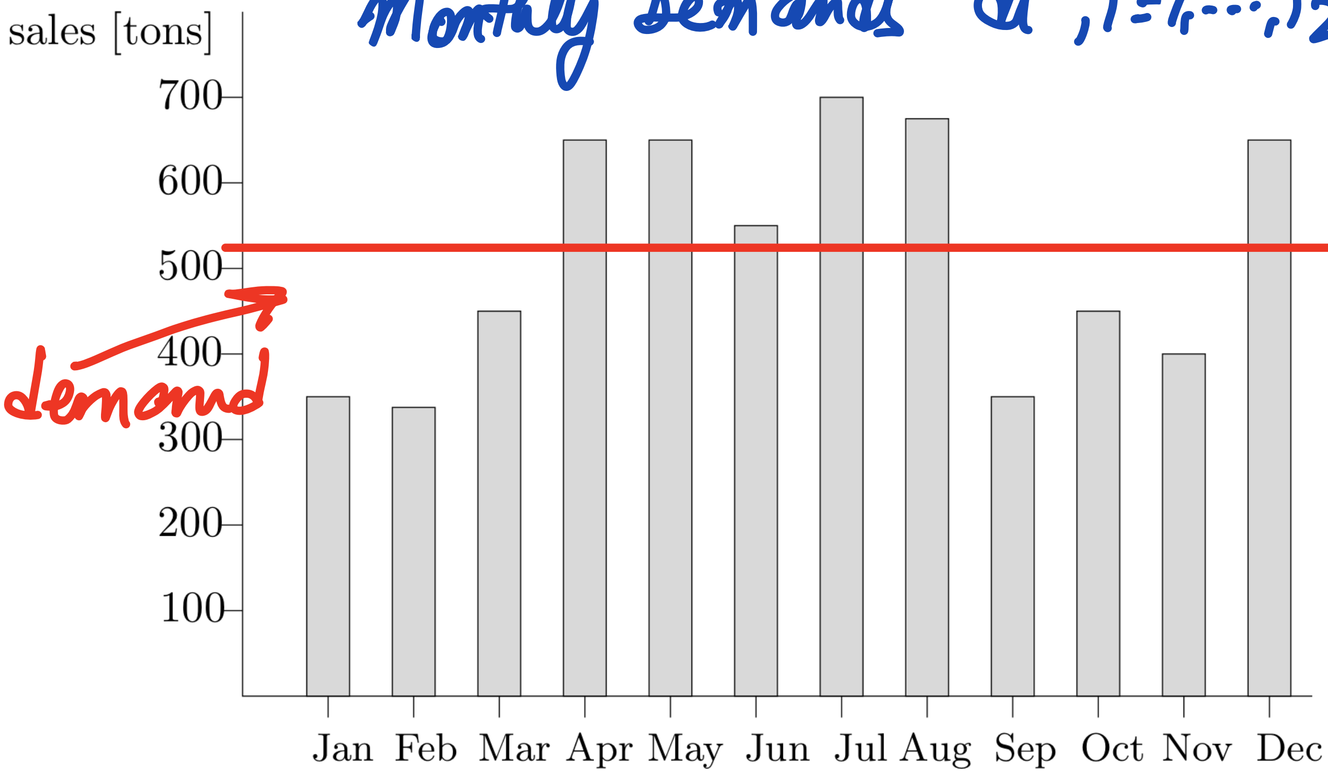


average demand

Production Planning

[MG] p.16

(1) x_i – production at month i

(2) s_i ~ surplus at end of month i

(3) $x_i + s_{i-1} = d_i + s_i$

$$\min \underbrace{\sum_{i=1}^{12} 50\,|x_i - x_{i-1}|}_{\text{cost of changing production}} + \underbrace{\sum_{i=1}^{12} 20\, s_i}_{\text{cost of storage}}$$

Production Planning

[MG] p.16

(1) x_i — production at month i

(2) s_i ~ surplus at end of month i

(3) $x_i + s_{i-1} = d_i + s_i$

$$\min \sum_{i=1}^{12} 50\,|x_i - x_{i-1}| + \sum_{i=1}^{12} 20\, s_i$$

$$x_i - x_{i-1} = y_i - z_i \;,\quad y_i, z_i \geq 0$$

$$|x_i - x_{i-1}| = y_i + z_i$$

Production Planning

[MG] p.16

$$\min \quad 50\sum_{i=1}^{12}(y_i+z_i) + 20\sum_{i=1}^{12} s_i$$

s.t.

$$x_i + s_{i-1} - s_i = d_i \qquad i=1,2,\cdots 12$$

$$x_i - x_{i-1} = y_i - z_i$$

$$x_0 = 0$$

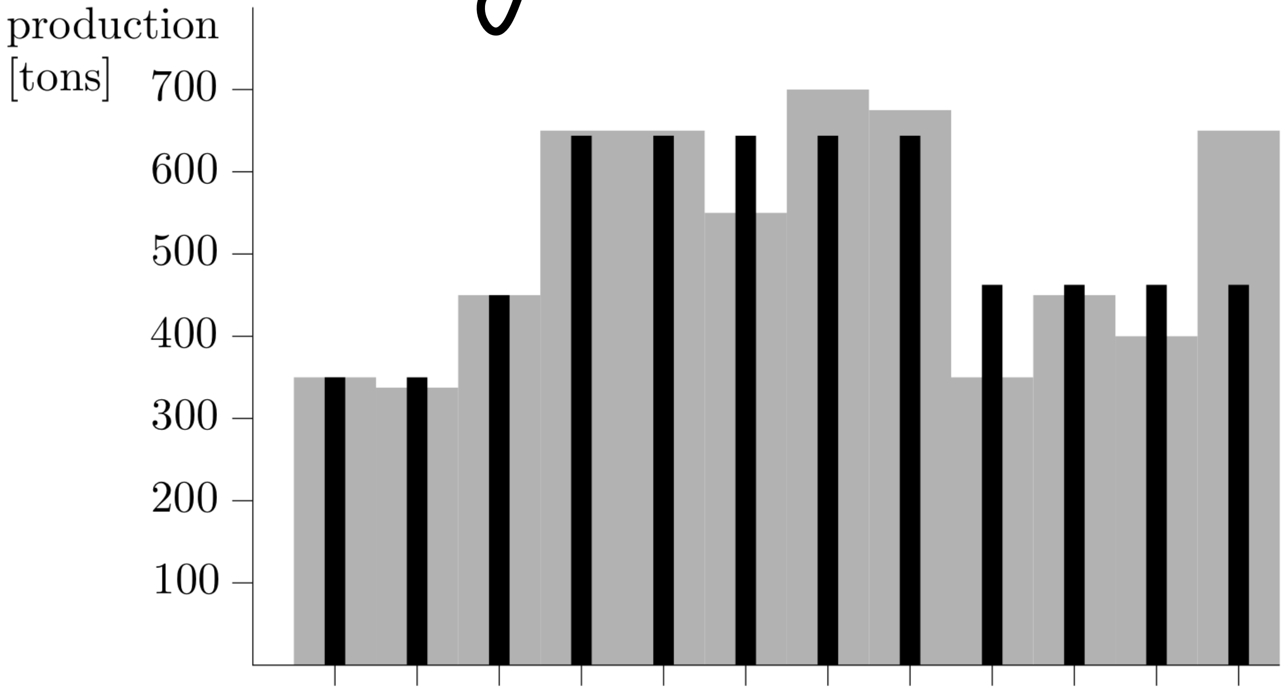
$$s_0 = 0$$

$$s_{12} = 0$$

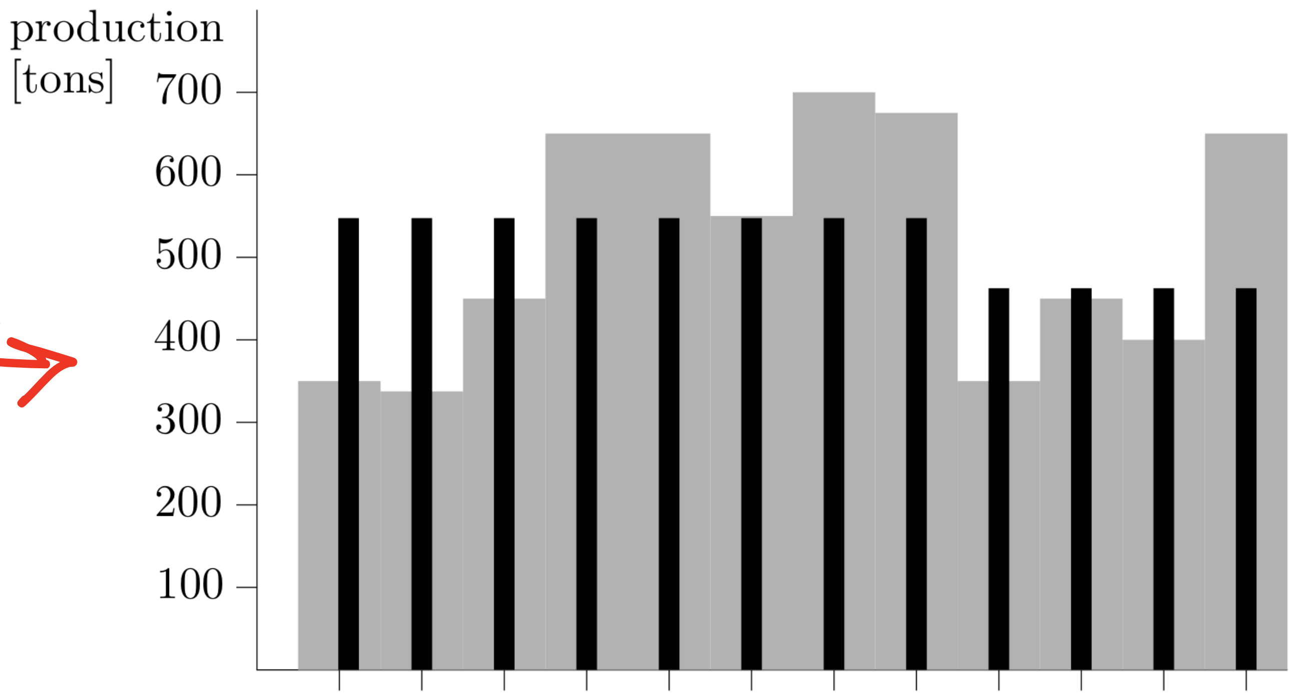
$$x_i,\ y_i,\ z_i,\ s_i \geq 0 \qquad i=1,2,\cdots,12$$

Production Planning

[MG] p.16



no storage cost



[V] p. 209

Jan	390	May	310	Sep	550
Feb	420	Jun	590	Oct	360
Mar	340	Jul	340	Nov	420
Apr	320	Aug	580	Dec	600

TABLE 12.2. Projected labor hours by month.

12.7 *Sales Force Planning*. A distributor of office equipment finds that the business has seasonal peaks and valleys. The company uses two types of sales persons: (a) regular employees who are employed year-round and cost the company \$17.50/h (fully loaded for benefits and taxes) and (b) temporary employees supplied by an outside agency at a cost of \$25/h. Projections for the number of hours of labor by month for the following year are shown in Table 12.2. Let a_i denote the number of hours of labor needed for month i and let x denote the number of hours per month of labor that will be handled by regular employees. To minimize total labor costs, one needs to solve the following optimization problem:

$$\text{minimize} \sum_i (25 \max(a_i - x, 0) + 17.50x).$$

(a) Show how to reformulate this problem as a linear programming problem.
(b) Solve the problem for the specific data given above.
(c) Use calculus to find a formula giving the optimal value for x.

$$\min \quad 25 \sum_{i=1}^{12} y_i + (17.5 \times 12)\, x$$

$$\text{s.t.} \quad \left.\begin{aligned} a_i - x &\le y_i \\ 0 &\le y_i \end{aligned}\right\} \quad \max(a_i - x, 0) \le y_i$$

$$x \ge 0$$

Portfolio Selection (Markowitz Model)

[V] Ch. 13

① R_i — (annual) return of i^{th} investment
(a random Variable)

② x_j — proportion of investment in i
($x_j \geq 0$, $\sum x_j = 1$)

$$\max \sum_j x_j \underbrace{R_j}_{\text{random}}$$

or

$$\max \sum_j x_j \underbrace{E R_j}_{\text{average return}}$$

Portfolio Selection (Markowitz Model)

[v] Ch. 13

① R_i — (annual) return of i^{th} investment
(a random Variable)

② x_j — proportion of investment in i
($x_j \geq 0$, $\sum x_j = 1$)

$$\max \quad \mu \sum_j x_j ER_j - E \left| \sum_j x_j \tilde{R}_j \right|$$

$$\tilde{R}_j = \underbrace{R_j - ER_j}_{\text{fluctuation}}$$

Portfolio Selection (Markowitz Model)

[V] Ch. 13

Estimation of $E R_j$ and $E\left|\sum_j x_j \tilde{R}_j\right|$

Given historical data, $R_j(t)$, $t = 1, \ldots, T$

① $E R_j = \frac{1}{T} \sum_{t=1}^{T} R_j(t) \quad (= r_j)$

② $E\left|\sum_j x_j \tilde{R}_j\right| = \frac{1}{T} \sum_{t=1}^{T} \left|\sum_j x_j \left(R_j(t) - r_j\right)\right|$

Portfolio Selection (Markowitz Model)

[V] Ch. 13

$$\max \ \mu \sum_j x_j r_j - \frac{1}{T} \sum_{t=1}^{T} \left| \sum_j x_j \left(R_j(t) - r_j \right) \right|$$

$$\text{s.t.} \quad \sum_j x_j = 1, \quad x_j \geq 0, \qquad \left(r_j = \frac{1}{T} \sum_{t=1}^{T} R_j(t) \right)$$

(LP)

$$\boxed{\begin{aligned} \max \ & \mu \sum_j x_j r_j - \frac{1}{T} \sum_{t=1}^{T} y_t \\ \text{s.t.} \ & -y(t) \leq \sum_j x_j \left(R_j(t) - r_j \right) \leq y(t) \\ & \sum_j x_j = 1, \quad x_j \geq 0 \end{aligned}}$$

Portfolio Selection (Markowitz Model)

[V] Ch.24

$$\max \quad \mu \sum_j x_j E R_j - \underbrace{E\left(\sum_j x_j \tilde{R}_j\right)^2}_{\text{variance}}$$

$$\text{s.t.} \quad \sum_j x_j = 1, \quad x_j \geq 0$$

$$E\left(\sum_j x_j \tilde{R}_j\right)^2 = E\left(\sum_{i,j} x_i x_j \tilde{R}_i \tilde{R}_j\right)$$

$$= \sum_{i,j} x_i x_j \, E(\tilde{R}_i \tilde{R}_j)$$

Portfolio Selection (Markowitz Model)

[v] ch.24

$$\max \quad \mu \sum_j x_j \underbrace{ER_j}_{r_j} - \underbrace{E\Big(\sum_j x_j \tilde{R}_j\Big)^2}_{\text{variance}}$$

$$\text{s.t.} \quad \sum_j x_j = 1, \quad x_j \geq 0$$

$$E\Big(\sum_j x_j \tilde{R}_j\Big)^2 = E\Big(\sum_{i,j} x_i x_j \tilde{R}_i \tilde{R}_j\Big)$$

$$= \sum_{i,j} x_i x_j \underbrace{E(\tilde{R}_i \tilde{R}_j)}_{C_{ij}}$$

Portfolio Selection (Markowitz Model)

[V] Ch. 24

$$\boxed{\begin{array}{l} \max \quad \mu \sum_j x_j r_j - \sum_{i,j} x_i x_j C_{ij} \\ \text{s.t.} \quad \sum_j x_j = 1, \quad x_j \geq 0 \end{array}}$$

Quadratic fct

Portfolio Selection (Markowitz Model)

[v] Ch. 24

$$\max \quad \mu \sum_j x_j r_j - \sum_{i,j} x_i x_j C_{ij}$$

$$\text{s.t.} \quad \sum_j x_j = 1, \quad x_j \geq 0$$

Quadratic fct

Quadratic Programming

$$\min \quad c^T x + \frac{1}{2} x^T Q x$$

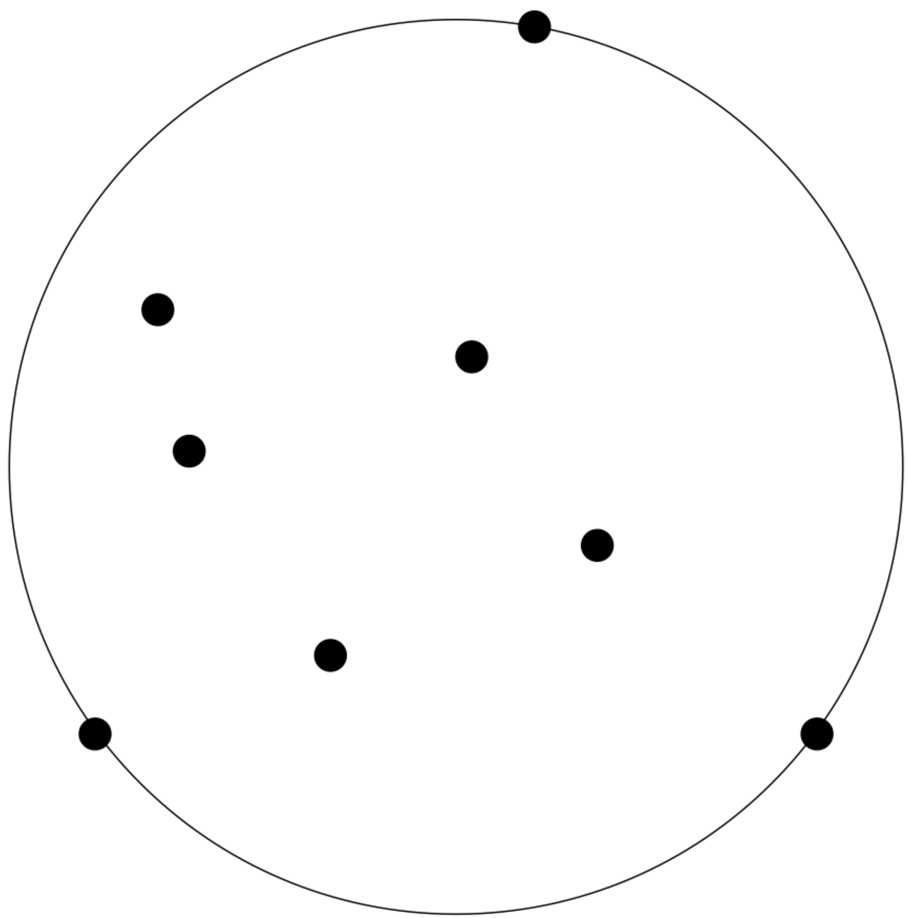
$$\text{s.t.} \quad Ax \leq b$$

$$x \geq 0$$

[MG] p. 184

8.7 Smallest Balls and Convex Programming

The smallest ball problem. We are given points $\mathbf{p}_1, \dots, \mathbf{p}_n \in \mathbb{R}^d$, and we want to find a ball of the smallest radius that contains all the points.[8]



[MG] p. 184

8.7.4 Theorem. *Let $\mathbf{p}_1, \ldots, \mathbf{p}_n$ be points in $\mathbb{R}^d$, and let Q be the $d \times n$ matrix whose jth column is formed by the d coordinates of the point $\mathbf{p}_j$. Let us consider the optimization problem*

$$\begin{array}{ll} \text{minimize} & \mathbf{x}^T Q^T Q\mathbf{x} - \sum_{j=1}^n x_j \mathbf{p}_j^T \mathbf{p}_j \\ \text{subject to} & \sum_{j=1}^n x_j = 1 \\ & \mathbf{x} \geq \mathbf{0} \end{array} \tag{8.15}$$

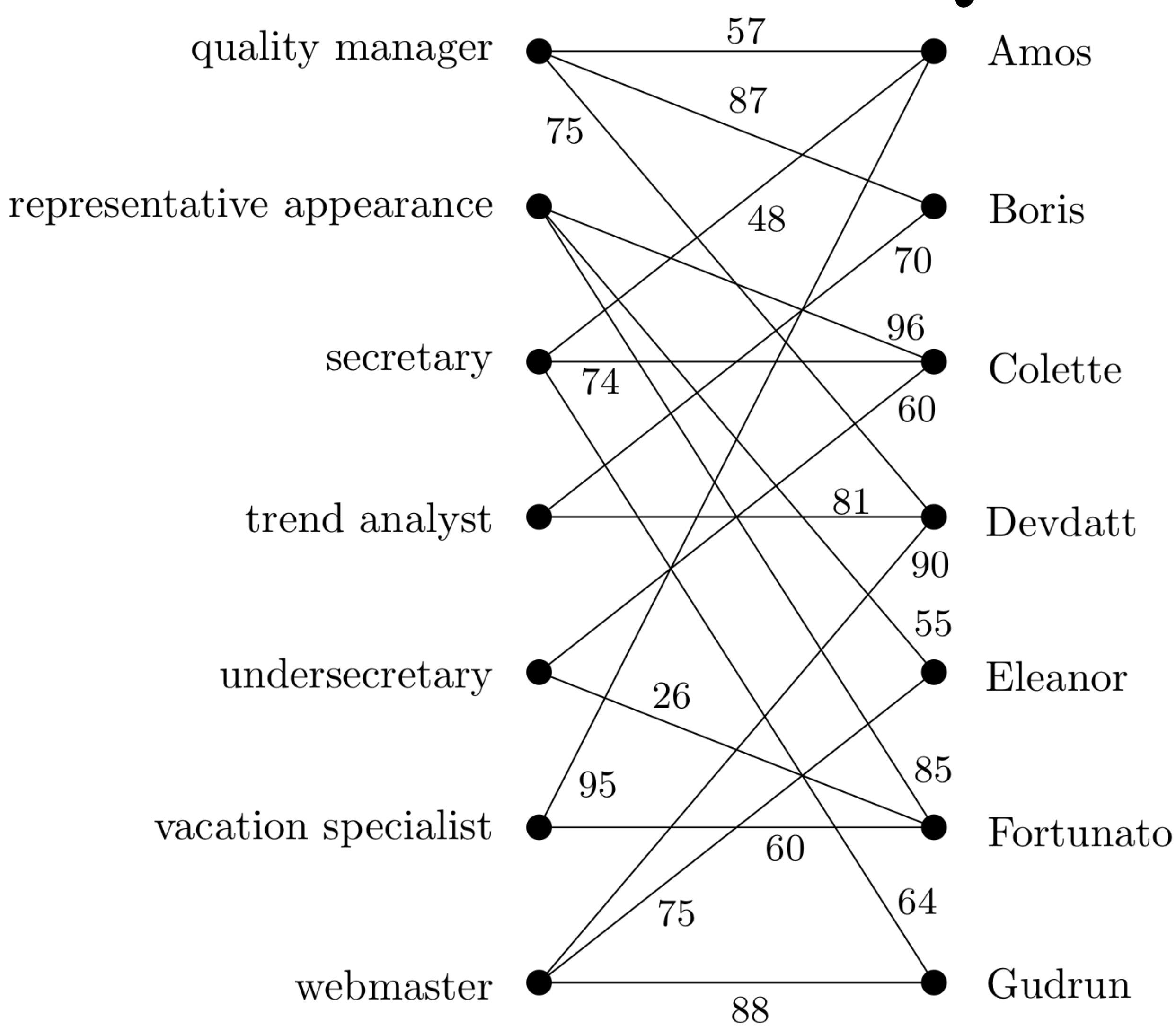
in the variables $x_1, \ldots, x_n$. Then the objective function $f(\mathbf{x}) := \mathbf{x}^T Q^T Q\mathbf{x} - \sum_{j=1}^n x_j \mathbf{p}_j^T \mathbf{p}_j$ is convex, and the following statements hold.

(i) *Problem (8.15) has an optimal solution $\mathbf{x}^*$.*

(ii) *There exists a point $\mathbf{p}^*$ such that $\mathbf{p}^* = Q\mathbf{x}^*$ holds for every optimal solution $\mathbf{x}^*$. Moreover, the ball with center $\mathbf{p}^*$ and squared radius $-f(\mathbf{x}^*)$ is the unique ball of smallest radius containing P.*

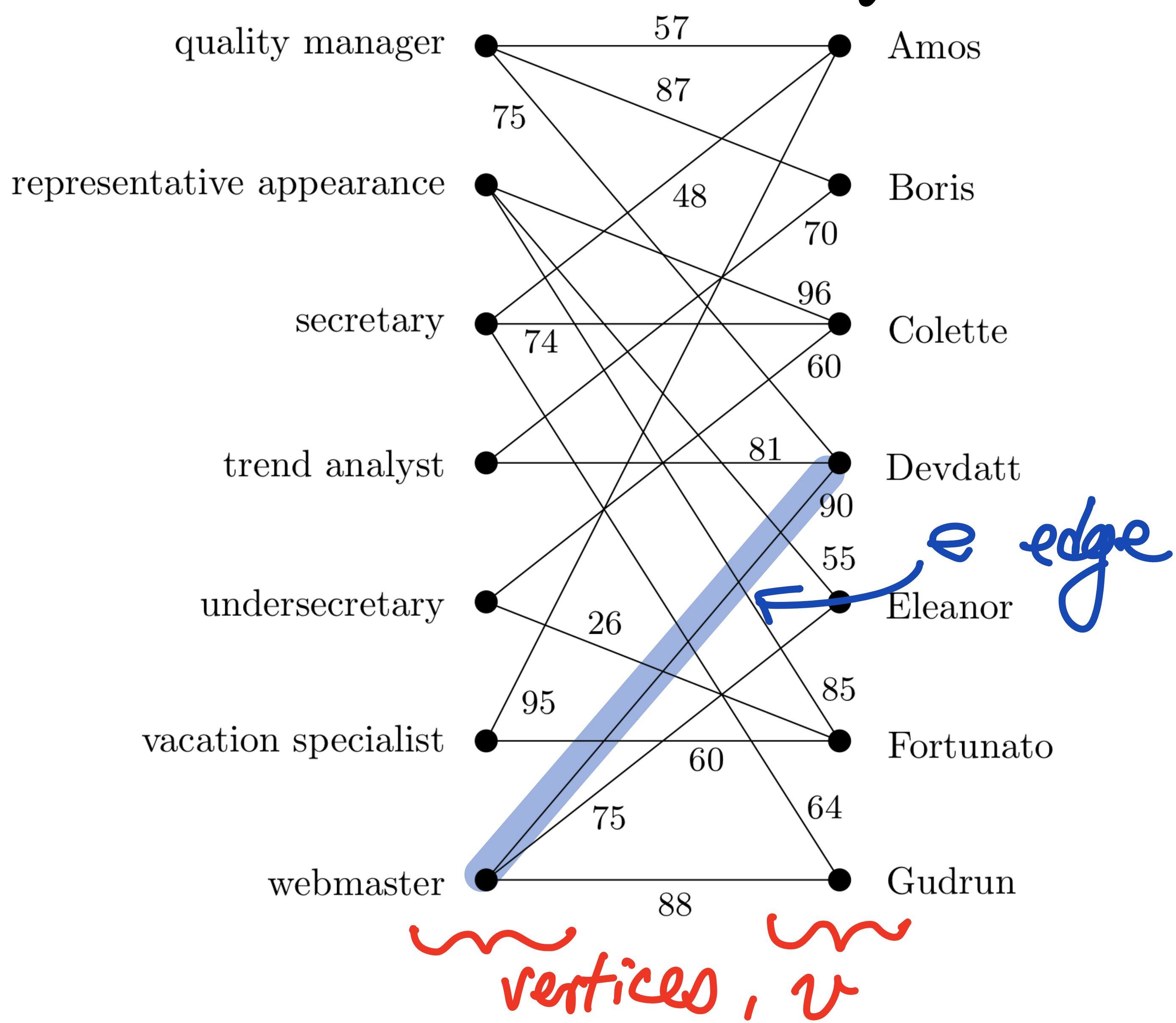
Integer Programming

[MG] p. 31 Maximum Weight Matching



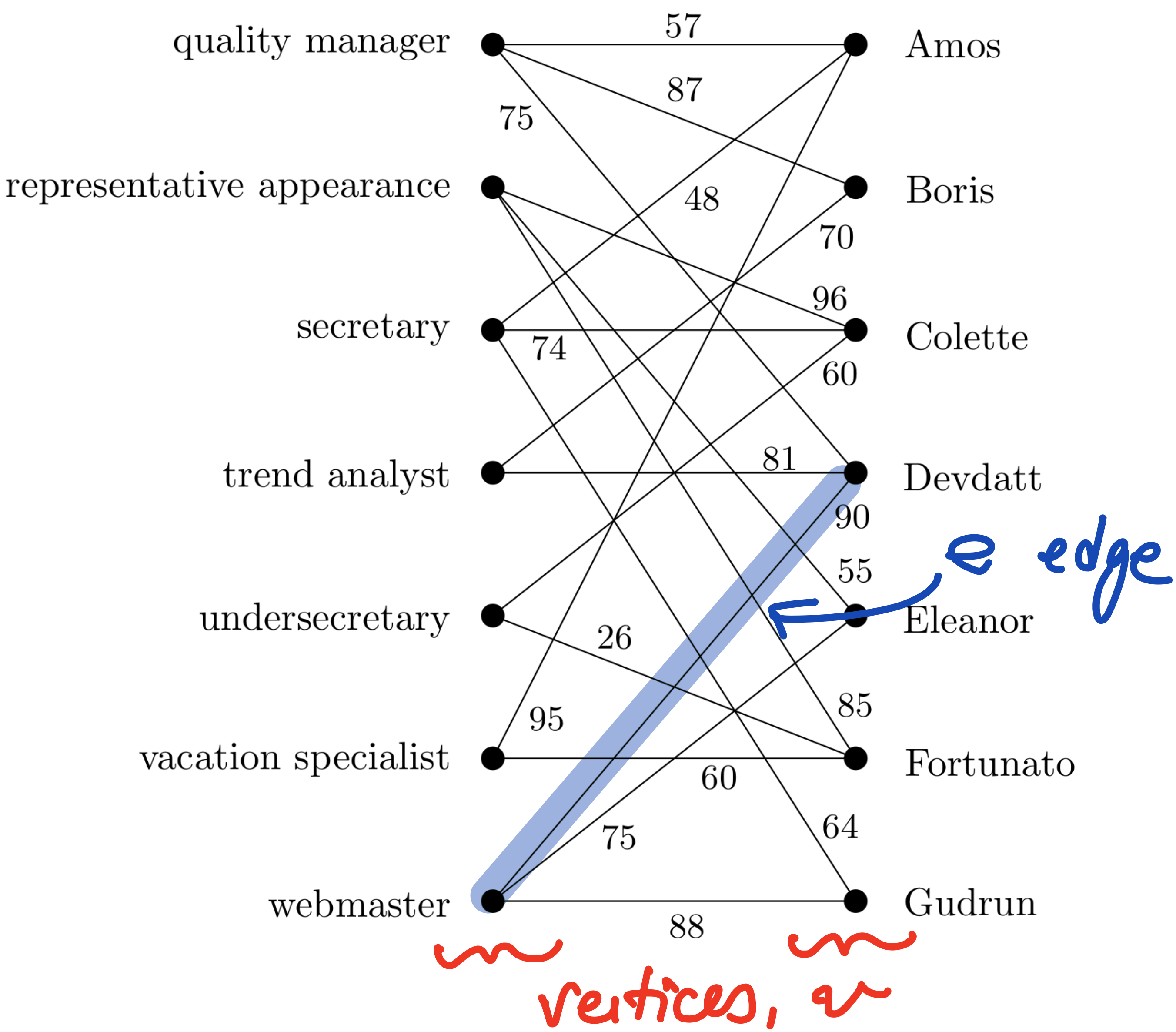
Integer Programming

[MG] p. 31 Maximum Weight Matching



Integer Programming

[MG] p. 31 Maximum Weight Matching



$$\max \sum_e w_e x_e$$

s.t.

$$\sum_{e,\, v \in e} x_e = 1$$

$$x_e \in \{0, 1\}$$

Integer Programming

[MG] p.148 Machine Scheduling

	Single B&W	Duplex B&W	Duplex Color
Master's thesis, 90 pages two-sided, 10 B&W copies	—	45 min	60 min
All the Best Deals flyer, 1 page one-sided, 10,000 B&W copies	2h 45 min	4h 10 min	5h 30 min
Buyer's Paradise flyer, 1 page one-sided, 10,000 B&W copies	2h 45 min	4h 10 min	5h 30 min
Obituary, 2 pages two-sided, 100 B&W copies	—	2 min	3 min
Party platform, 10 pages two-sided, 5,000 color copies	—	—	3h 30 min

Integer Programming

[MG] p. 148 Machine Scheduling

$M = \{1, 2, \ldots, m\}$ m - Machines

$J = \{1, 2, \ldots, n\}$ n - Jobs

d_{ij} = running time of Job j in Machine i

$$
\begin{array}{lll}
\min & t & \text{(total run time)} \\
\text{s.t.} & \sum_{i \in M} x_{ij} = 1 & j \in J \\
& \sum_{j \in J} d_{ij} x_{ij} \leq t & \\
& x_{ij} \in \{0, 1\} &
\end{array}
$$

Integer Programming

Knapsack Problem

[V]
p.413

23.1 *Knapsack Problem*. Consider a picnicker who will be carrying a knapsack that holds a maximum amount b of "stuff." Suppose that our picnicker must decide what to take and what to leave behind. The jth thing that might be taken occupies a_j units of space in the knapsack and will bring c_j amount of "enjoyment." The knapsack problem then is to maximize enjoyment subject to the constraint that the stuff brought must fit into the knapsack:

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{j=1}^{n} c_j x_j \\ \text{subject to} & \displaystyle\sum_{j=1}^{n} a_j x_j \leq b \\ & \qquad x_j \in \{0,1\} \qquad j = 1, 2, \ldots, n. \end{array}$$

[C]
p.201

In the usual notation, the knapsack problem is recorded as

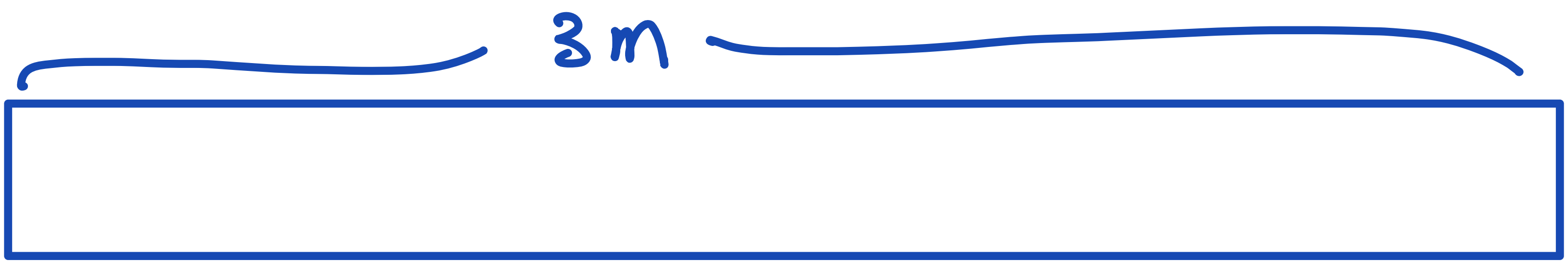
$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{i=1}^{m} c_i x_i \\ \text{subject to} & \displaystyle\sum_{i=1}^{m} a_i x_i \leq b \\ & \qquad x_i = \text{nonnegative integer} \qquad (i = 1, 2, \ldots, m). \end{array} \tag{13.5}$$

Cutting Stock Problem [MG p. 26]

A paper mill manufactures rolls of paper of a standard width 3 meters. But customers want to buy paper rolls of shorter width, and the mill has to cut such rolls from the 3 m rolls. One 3 m roll can be cut, for instance, into two rolls 93 cm wide, one roll of width 108 cm, and a rest of 6 cm (which goes to waste).

Let us consider an order of

- 97 rolls of width 135 cm,
- 610 rolls of width 108 cm,
- 395 rolls of width 93 cm, and
- 211 rolls of width 42 cm.



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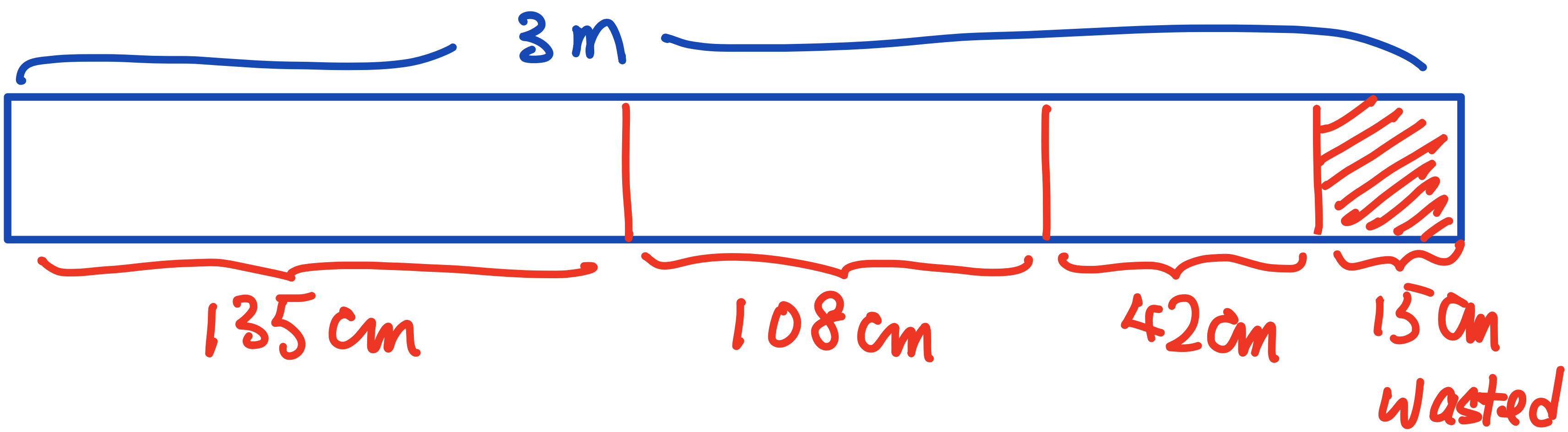
e.g.



Cutting Stock Problem

In order to engage linear programming one has to be generous in introducing variables. We write down all of the requested widths: 135 cm, 108 cm, 93 cm, and 42 cm. Then we list all possibilities of cutting a 3 m paper roll into rolls of some of these widths (we need to consider only possibilities for which the wasted piece is shorter than 42 cm):

P1:	2×135	P7:	$108 + 93 + 2 \times 42$
P2:	$135 + 108 + 42$	P8:	$108 + 4 \times 42$
P3:	$135 + 93 + 42$	P9:	3×93
P4:	$135 + 3 \times 42$	P10:	$2 \times 93 + 2 \times 42$
P5:	$2 \times 108 + 2 \times 42$	P11:	$93 + 4 \times 42$
P6:	$108 + 2 \times 93$	P12:	7×42

$$258 \leq 135 n_1 + 108 n_2 + 93 n_3 + 42 n_4 \leq 300$$

$0 \leq n_i$, integers

Cutting Stock Problem

In order to engage linear programming one has to be generous in introducing variables. We write down all of the requested widths: 135 cm, 108 cm, 93 cm, and 42 cm. Then we list all possibilities of cutting a 3 m paper roll into rolls of some of these widths (we need to consider only possibilities for which the wasted piece is shorter than 42 cm):

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P4:	$135 + 3 \times 42$	P10:	$2 \times 93 + 2 \times 42$
P5:	$2 \times 108 + 2 \times 42$	P11:	$93 + 4 \times 42$
P6:	$108 + 2 \times 93$	P12:	7×42

$x_i = \#$ of P_i, $i = 1, \ldots, 12$

$x_i \geq 0$, integers

min $x_1 + x_2 + \cdots + x_{12}$

Cutting Stock Problem

In order to engage linear programming one has to be generous in introducing variables. We write down all of the requested widths: 135 cm, 108 cm, 93 cm, and 42 cm. Then we list all possibilities of cutting a 3 m paper roll into rolls of some of these widths (we need to consider only possibilities for which the wasted piece is shorter than 42 cm):

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P3:	$135 + 93 + 42$	P9:	3×93
P4:	$135 + 3 \times 42$	P10:	$2 \times 93 + 2 \times 42$
P5:	$2 \times 108 + 2 \times 42$	P11:	$93 + 4 \times 42$
P6:	$108 + 2 \times 93$	P12:	7×42

$$2x_1 + x_2 + x_3 + x_4 \geq 97$$

$$x_2 + 2x_5 + x_6 + x_7 + x_8 \geq 610$$

$$x_3 + 2x_6 + x_7 + 2x_9 + x_{11} \geq 395$$

$$x_2 + x_3 + x_4 + 2x_5 + 2x_7 + 4x_8 + 2x_{10} + 4x_{11} + 7x_{12} \geq 211$$

Sparse Solutions of Linear System [MG, p.167]

Sparse solution of underdetermined system of linear equations

Given an $m \times n$ matrix A with $m < n$, a vector $\mathbf{b} \in \mathbb{R}^m$, and an integer r, find an $\mathbf{x} \in \mathbb{R}^n$ such that

$$A\mathbf{x} = \mathbf{b} \quad \text{and} \quad |\mathrm{supp}(\mathbf{x})| \leq r \tag{8.10}$$

$\#\{i : x_i \neq 0\}$

if one exists.

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$\#\{i : x_i \neq 0\}$

if one exists.

Recovery of signals with noise:

$W \in \mathbb{R}^k \Longrightarrow$ encode $z = QW$

$\Longrightarrow$ transmit $\tilde{z} = z + X$

noise: only a few zeros

Given $\tilde{z}$, how to recover z?

$$\tilde{z} = QW + X$$

$$A\tilde{z} = \cancel{AQW} + AX$$

Sparse Solutions of Linear System [MG, p.167]

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if one exists.

$$\underbrace{A\tilde{z}}_{b \text{ given}} = \cancel{AQW} + AX \implies AX = b$$

$\boxed{\text{find } X \text{ with as few non-zeros as possible}}$

$$\tilde{z} = z + X \implies z = \tilde{z} - X$$

Sparse Solutions of Linear System [MG, p.167]

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if one exists.

Data Compression:

$$\underbrace{A\,X}_{\text{representation}} = \underbrace{b}_{\text{data}} \qquad A^{m\times n},\ b \in \mathbb{R}^m$$

$$x_1 U_1 + x_2 U_2 + \cdots + x_n U_n = b$$

coordinates of b w.r.t. cols of A

Sparse Solutions of Linear System [MG, p.167]

Sparse solution of underdetermined system of linear equations

Given an $m \times n$ matrix A with $m < n$, a vector $\mathbf{b} \in \mathbb{R}^m$, and an integer r, find an $\mathbf{x} \in \mathbb{R}^n$ such that

$$A\mathbf{x} = \mathbf{b} \quad \text{and} \quad \underbrace{|\mathrm{supp}(\mathbf{x})|}_{\#\{i \,:\, x_i \neq 0\}} \leq r \tag{8.10}$$

if one exists.

$$\min \|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

$$\text{s.t.} \quad Ax = b$$

Sparse Solutions of Linear System [MG, p.167]

Sparse solution of underdetermined system of linear equations

Given an $m \times n$ matrix A with $m < n$, a vector $\mathbf{b} \in \mathbb{R}^m$, and an integer r, find an $\mathbf{x} \in \mathbb{R}^n$ such that

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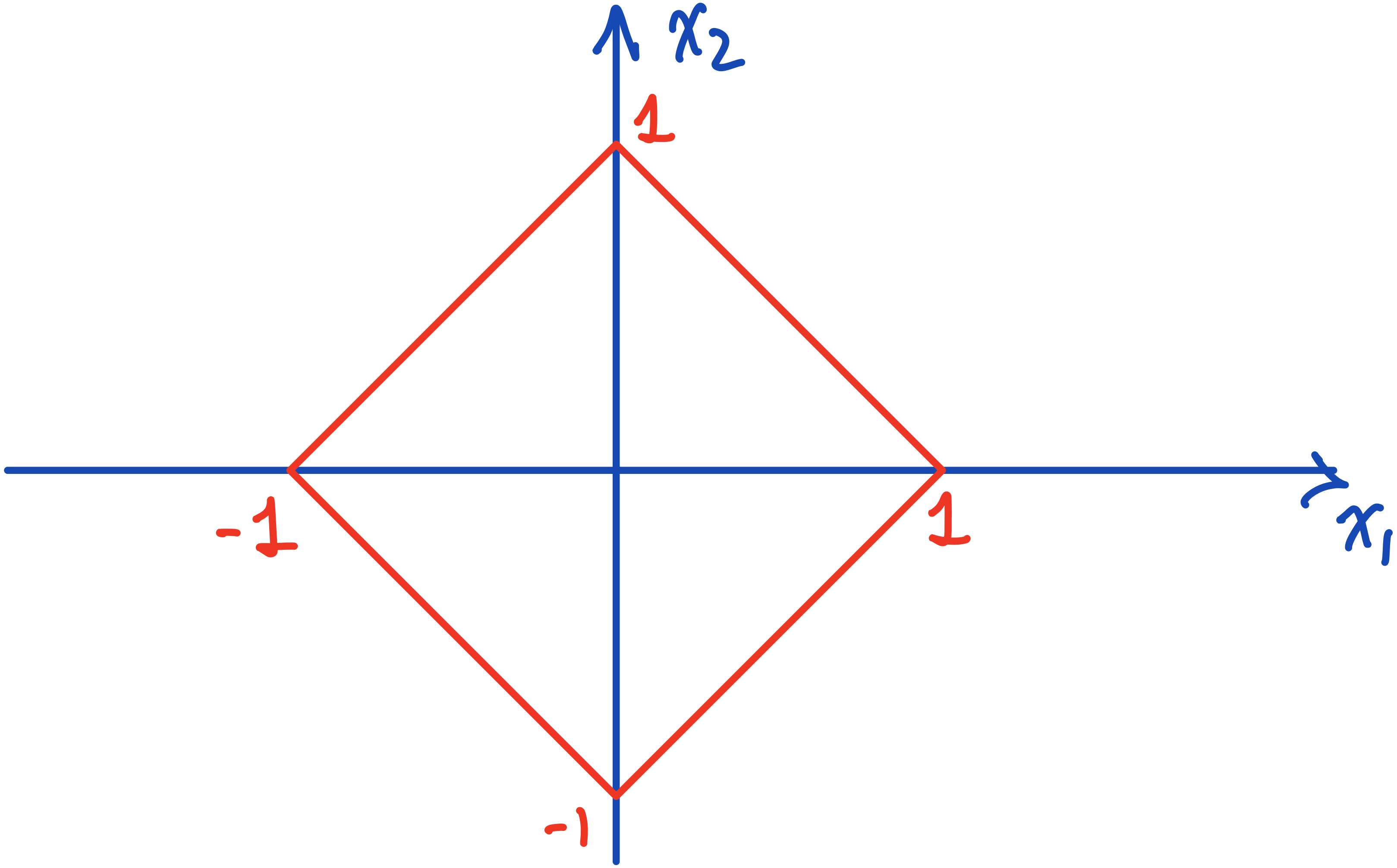
if one exists.

$$\min \|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$
$$\text{s.t. } Ax = b$$

$$\left(\begin{array}{ll} \min & u_1 + u_2 + \cdots + u_n \\ \text{s.t.} & -u_i \leq x_i \leq u_i, \quad i = 1, 2, \ldots, n \\ & Ax = b \end{array}\right)$$

Sparse Solutions of Linear System [MG, p.167]

$n=2 \qquad \|x\|_1 = |x_1| + |x_2| = 1$



Sparse Solutions of Linear System [MG, p.167]

$$\min \|x\|_1 \quad \text{s.t.} \quad Ax = b$$