Farkas Lemma

LEMMA 10.5. *The system $Ax \leq b$ has no solutions if and only if there is a y such that*

$$
\begin{aligned}
A^T y &= 0 \\
y &\geq 0 \qquad (10.8) \\
b^T y &< 0.
\end{aligned}
$$

PROOF. Consider the linear program

(P)

$$
\begin{array}{ll}
\text{maximize} & 0 \\
\text{subject to} & Ax \leq b
\end{array}
$$

and its dual

(D)

$$
\begin{array}{lrl}
\text{minimize} & b^T y & \\
\text{subject to} & A^T y &= 0 \\
& y &\geq 0.
\end{array}
$$

Farkas Lemma (Other Versions)

6.4.1 Proposition (Farkas lemma). *Let A be a real matrix with m rows and n columns, and let $\mathbf{b} \in \mathbb{R}^m$ be a vector. Then exactly one of the following two possibilities occurs:*

(F1) *There exists a vector $\mathbf{x} \in \mathbb{R}^n$ satisfying $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.*

(F2) *There exists a vector $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y}^T A \geq \mathbf{0}^T$ and $\mathbf{y}^T \mathbf{b} < 0$.*

alternatives

[MG] # Farkas Lemma (Other Versions)

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alternatives

Fredholm Alternatives

$$AX = b$$

(F1) $AX = b$ has a solution

(F2) There is y s.t. $y^T A = 0$ and $y^T b \neq 0$

[MG] Farkas Lemma (Other Versions)

6.4.3 Proposition (Farkas lemma in three variants). *Let A be a real matrix with m rows and n columns, and let $\mathbf{b} \in \mathbb{R}^m$ be a vector.*

(i) *The system $A\mathbf{x} = \mathbf{b}$ has a nonnegative solution if and only if every $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y}^T A \geq \mathbf{0}^T$ also satisfies $\mathbf{y}^T\mathbf{b} \geq \mathbf{0}$.*

(ii) *The system $A\mathbf{x} \leq \mathbf{b}$ has a nonnegative solution if and only if every nonnegative $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y}^T A \geq \mathbf{0}^T$ also satisfies $\mathbf{y}^T\mathbf{b} \geq 0$.*

(iii) *The system $A\mathbf{x} \leq \mathbf{b}$ has a solution if and only if every nonnegative $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{y}^T A = \mathbf{0}^T$ also satisfies $\mathbf{y}^T\mathbf{b} \geq 0$.*

(in fact, (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii))

	The system $A\mathbf{x} \leq \mathbf{b}$	The system $A\mathbf{x} = \mathbf{b}$
has a solution $\mathbf{x} \geq \mathbf{0}$ iff	$\mathbf{y} \geq \mathbf{0}, \mathbf{y}^T A \geq \mathbf{0}$ $\Rightarrow \mathbf{y}^T\mathbf{b} \geq 0$	$\mathbf{y}^T A \geq \mathbf{0}^T$ $\Rightarrow \mathbf{y}^T\mathbf{b} \geq 0$
has a solution $\mathbf{x} \in \mathbb{R}^n$ iff	$\mathbf{y} \geq \mathbf{0}, \mathbf{y}^T A = \mathbf{0}$ $\Rightarrow \mathbf{y}^T\mathbf{b} \geq 0$	$\mathbf{y}^T A = \mathbf{0}^T$ $\Rightarrow \mathbf{y}^T\mathbf{b} = 0$

Fredholm Alt.

[MG] Farkas Lemma (Other Versions)

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(in fact, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii))

$\sim$(i) $Ax = b$, $x \geq 0$ has not solution
iff there is y s.t. $y^TA \geq 0^T$, $y^Tb < 0$

[MG] Farkas Lemma (Other Versions)

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(in fact, (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii))

$\sim$(ii) $AX \leq b$, $X \geq 0$ has no solution

iff there is $y \geq 0$ s.t. $y^TA \geq 0^T$, $y^Tb < 0$

[MG] Farkas Lemma (Other Versions)

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(in fact, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii))

$\sim$(iii) $Ax \leq b$ has no solution

iff there is y s.t. $y^TA = 0^T$, $y^Tb < 0$

[MG] Farkas Lemma (Other Versions)

6.4.1 Proposition (Farkas lemma). *Let A be a real matrix with m rows and n columns, and let $\mathbf{b} \in \mathbb{R}^m$ be a vector. Then exactly one of the following two possibilities occurs:*

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alternatives

[MG] Farkas Lemma (Other Versions)

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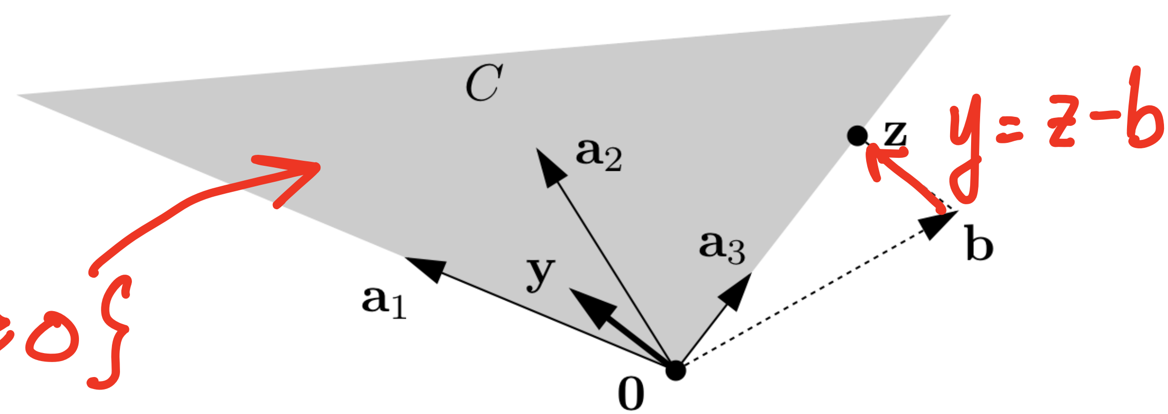
alternatives

The plan of the proof is straightforward: We let $\mathbf{z}$ be the point of C nearest to $\mathbf{b}$ (in the Euclidean distance), and we check that the vector $\mathbf{y} = \mathbf{z} - \mathbf{b}$ is as required; see the following illustration:



Cone: $\{AX \geq 0\}$

Farkas Lemma (Other Versions)

[C] P.248 #16.10

Derive the following theorems (with the vector inequality $\mathbf{v} > \mathbf{w}$ meaning, as usual, $v_k > w_k$ for all k) from the result of problem 16.9.

(i) P. Gordan (1873): The system $\mathbf{Ax} < \mathbf{0}$ is unsolvable if and only if the system $\mathbf{yA} = \mathbf{0}$, $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$ is solvable.

(ii) J. Farkas (1902): The system $\mathbf{Ax} \leq \mathbf{0}$, $\mathbf{bx} > 0$ is unsolvable if and only if the system $\mathbf{yA} = \mathbf{b}$, $\mathbf{y} \geq \mathbf{0}$ is solvable.

(iii) E. Stiemke (1915): The system $\mathbf{Ax} = \mathbf{0}$, $\mathbf{x} > \mathbf{0}$ is unsolvable if and only if the system $\mathbf{yA} \geq \mathbf{0}$, $\mathbf{yA} \neq \mathbf{0}$ is solvable.

(iv) J. A. Ville (1938): The system $\mathbf{Ax} < \mathbf{0}$, $\mathbf{x} \geq \mathbf{0}$ is unsolvable if and only if the system $\mathbf{yA} \geq \mathbf{0}$, $\mathbf{y} \geq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$ is solvable.

(v) A. W. Tucker (1956): The system $\mathbf{Ax} \geq \mathbf{0}$, $\mathbf{x} \geq \mathbf{0}$ has no solution with $x_k > 0$ if and only if the system $\mathbf{yA} \leq \mathbf{0}$, $\mathbf{y} \geq \mathbf{0}$ has a solution with

$$\sum_{i=1}^{m} a_{ik} y_i < 0.$$