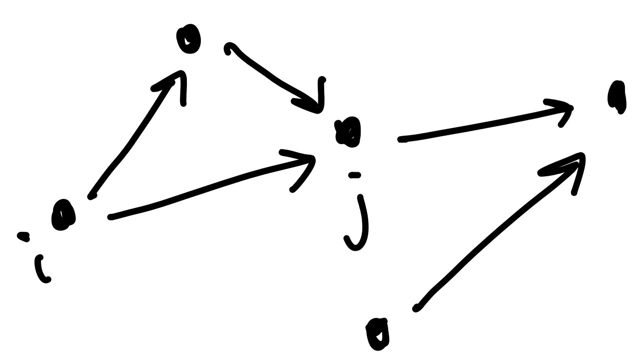
Network Flow ([V] Chapter 14)

$\mathcal{N}$ = set of nodes $\{i\}$

$\mathcal{A}$ = set of (directed) arcs

$\subseteq \{(i,j) : i,j \in \mathcal{N}\}$



Network = $(\mathcal{N}, \mathcal{A})$

graph (or digraph)

Network Flow ([V] Chapter 14)

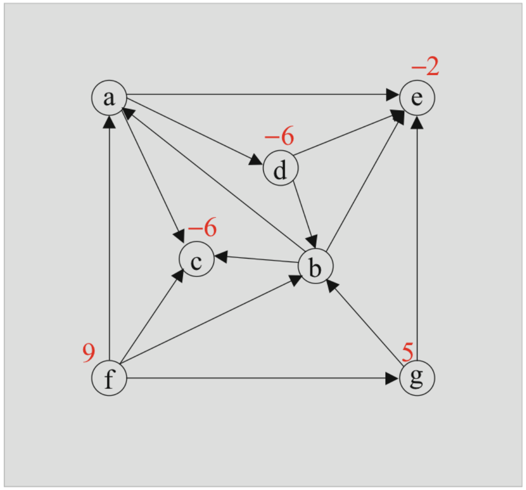


FIGURE 14.1. A network having 7 nodes and 14 arcs. The numbers written next to the nodes denote the supply at the node (negative values indicate demands; missing values indicate no supply or demand).

b_i

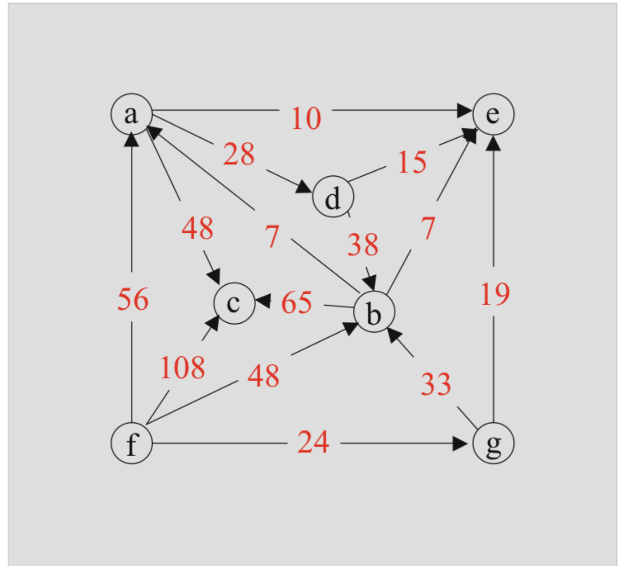


FIGURE 14.2. The costs on the arcs for the network in Figure 14.1.

c_{ij}

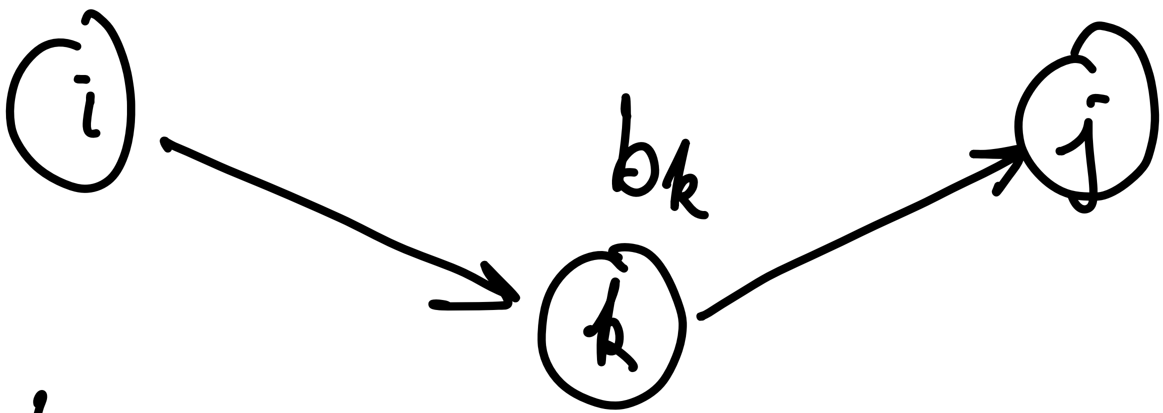
(1) $b_i > 0$ (supply) ; $b_i < 0$ (demand) $\quad \sum_{i \in N} b_i = 0$

(2) x_{ij} = amount transported from i to j

(3) c_{ij} = cost of transportation from i to j

(4) $\boxed{\min \sum_{(i,j) \in A} c_{ij} x_{ij}}$

Network Flow ([V] Chapter 14)

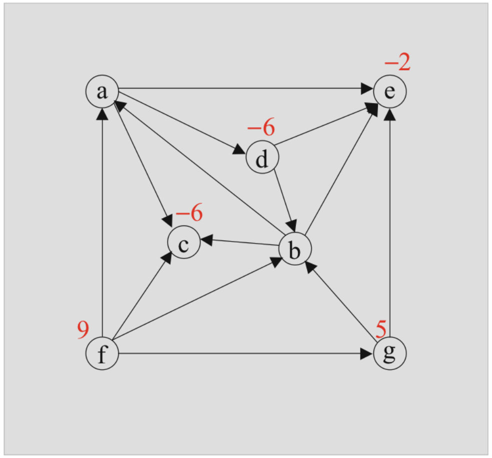


Balanced eqn at node k: $\sum_i x_{ik} + b_k = \sum_j x_{kj}$

$$\boxed{\sum_i x_{ik} - \sum_j x_{kj} = -b_k}$$

$$\boxed{\begin{array}{l} \min \quad c^T x \\ Ax = -b, \quad x \geq 0 \end{array}}$$

Network Flow ([V] Chapter 14)



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b_i

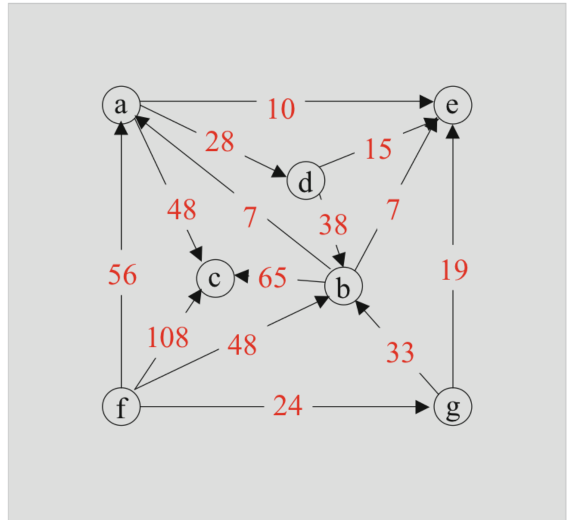


FIGURE 14.2. The costs on the arcs for the network in Figure 14.1.

c_{ij}

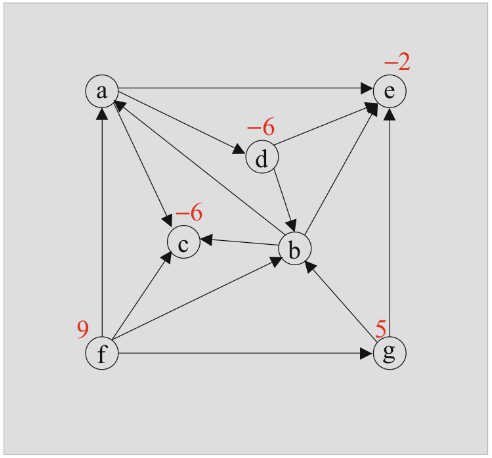
incidence matrix

$$x^T = \overbrace{\begin{bmatrix} x_{\rm ac} & x_{\rm ad} & x_{\rm ae} & x_{\rm ba} & x_{\rm bc} & x_{\rm be} & x_{\rm db} & x_{\rm de} & x_{\rm fa} & x_{\rm fb} & x_{\rm fc} & x_{\rm fg} & x_{\rm gb} & x_{\rm ge} \end{bmatrix}}^{\text{arcs}},$$

$$\begin{array}{c} (a) \\ (b) \\ (c) \\ (d) \\ (e) \\ (f) \\ (g) \end{array} \quad A = \begin{bmatrix} -1 & -1 & -1 & 1 & & & & & 1 & & & & & \\ & & & -1 & -1 & -1 & 1 & & & 1 & & & 1 & \\ 1 & & & & 1 & & & & & & 1 & & & \\ & 1 & & & & & -1 & -1 & & & & & & \\ & & 1 & & & 1 & & 1 & & & & & & 1 \\ & & & & & & & & -1 & -1 & -1 & -1 & & \\ & & & & & & & & & & & 1 & -1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ -6 \\ -6 \\ -2 \\ 9 \\ 5 \end{bmatrix}$$

$$c^T = \begin{bmatrix} 48 & 28 & 10 & 7 & 65 & 7 & 38 & 15 & 56 & 48 & 108 & 24 & 33 & 19 \end{bmatrix}.$$

Network Flow ([V] Chapter 14)



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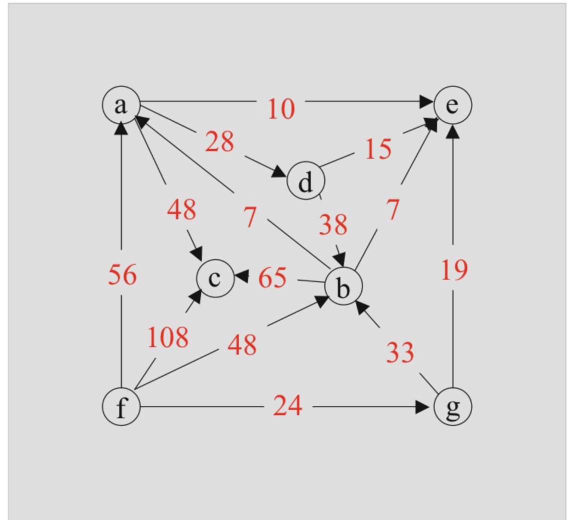
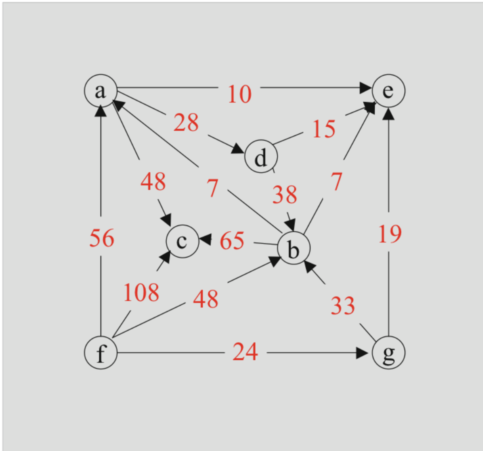


FIGURE 14.2. The costs on the arcs for the network in Figure 14.1.

$$A_{i,(k,l)} = \begin{cases} -1 & i = k \\ 1 & i = l \\ 0 & i \neq k,l \end{cases}$$

Network Flow ([V] Chapter 14)



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b_i

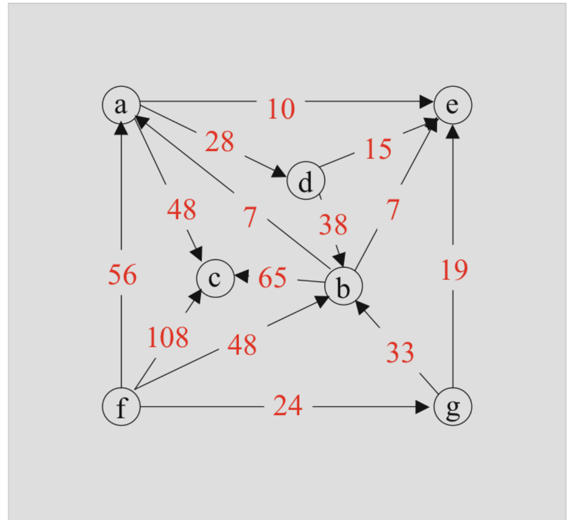


FIGURE 14.2. The costs on the arcs for the network in Figure 14.1.

c_{ij}

$\text{Rank}(A) = m-1$

- Sum of all rows $= 0$ $(\text{Rank}(A) \leq m-1)$
- Can delete one row (Root node)

Network Flow ([V] Chapter 14)

(P)
$$\boxed{\begin{aligned} &\min \quad c^T x \\ &Ax = -b, \quad x \geq 0 \end{aligned}}$$

(D)
$$\boxed{\begin{aligned} &\max \quad -b^T y \\ &\text{s.t.} \quad A^T y \leq c \end{aligned}}$$

$$A^T y + z = c, \quad z \geq 0$$

Network Flow ([V] Chapter 14)

(P)
$$\boxed{\begin{array}{l} \min \quad c^T x \\ Ax = -b, \quad x \geq 0 \end{array}}$$

(D)
$$\boxed{\begin{array}{l} \max \quad -b^T y \\ \text{s.t.} \quad A^T y + z = c, \quad z \geq 0 \end{array}}$$

$$\max \; -\sum_{i \in N} b_i y_i$$

$$\text{s.t.} \quad y_j - y_i + z_{ij} = c_{ij}, \quad (i,j) \in A$$
$$z_{ij} \geq 0$$

Network Flow ([V] Chapter 14)

(P)
$$\boxed{\begin{array}{l} \min \quad c^T x \\ Ax = -b, \quad x \geq 0 \end{array}}$$

(D)
$$\boxed{\begin{array}{l} \max \quad -b^T y \\ \text{s.t.} \quad A^T y + z = c, \quad z \geq 0 \end{array}}$$

$$\max \; -\sum_{i \in N} b_i y_i$$

$$\text{s.t.} \quad \underline{y_j - y_i} + z_{ij} = c_{ij}, \quad (i,j) \in A$$

$$z_{ij} \geq 0$$

$y_j \longrightarrow y_j + \text{const.}$

Spanning Trees and Bases

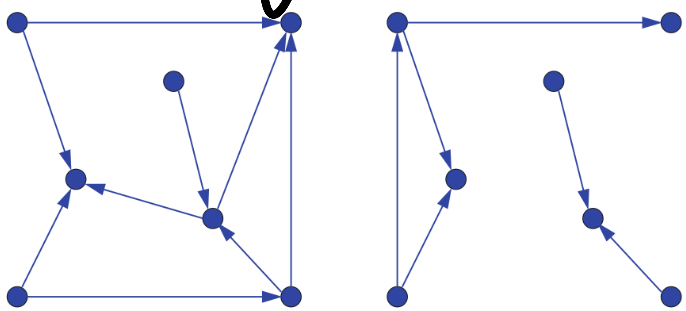


FIGURE 14.3. The network on the left is connected, whereas the one on the right is not.

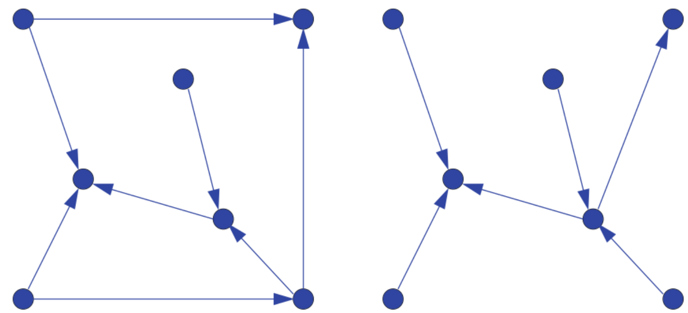


FIGURE 14.4. The network on the left contains a cycle, whereas the one on the right is acyclic.

- Path $= (n_1, n_2, \dots, n_k)$,
  for all l, (n_l, n_{l+1}) or $(n_{l+1}, n_l) \in \mathcal{A}$
- Cycle : $(n_1, n_2, \dots, n_k)$, $n_k = n_1$
- Tree : connected, acyclic
- Spanning tree $T = (\tilde{N}, \hat{A})$, tree and $\tilde{N} = N$

Spanning Trees and Bases

- Balanced Flow : $AX = -b$
- Feasible Flow: $X \geqslant 0$
- Tree Solution: $X_{(i,j)} = 0$ if $(i,j) \in$ Tree

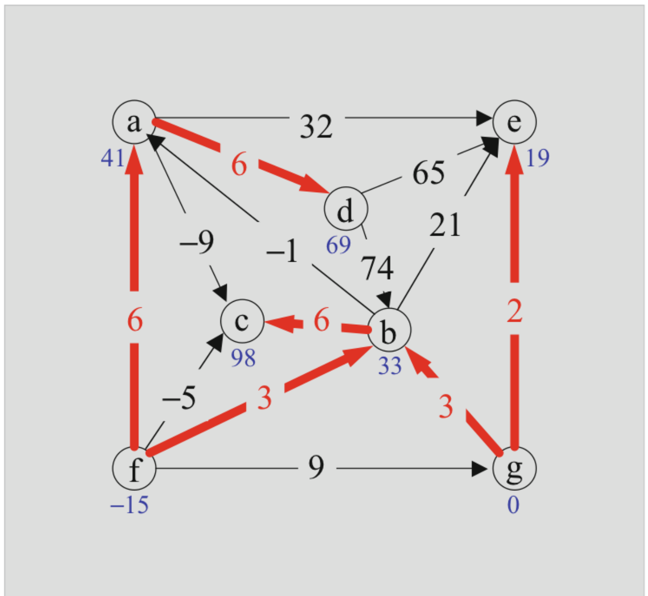
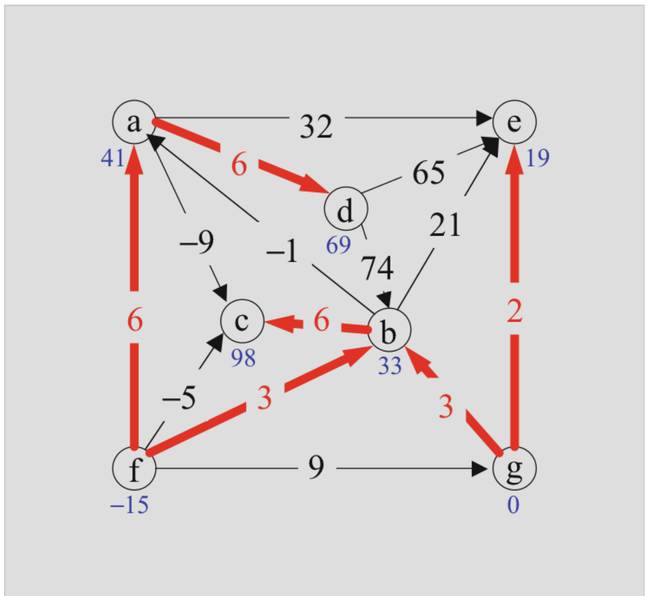


FIGURE 14.6. The fat arcs show a spanning tree for the network in Figure 14.1. The numbers shown on the arcs of the spanning tree are the primal flows, the numbers shown next to the nodes are the dual variables, and the numbers shown on the arcs not belonging to the spanning tree are the dual slacks.

Spanning Trees and Bases

- Balanced Flow : $AX = -b$
- Feasible Flow : $X \geq 0$
- Tree Solution : $X(i,j) = 0$ if $(i,j) \in$ Tree

might not be feasible



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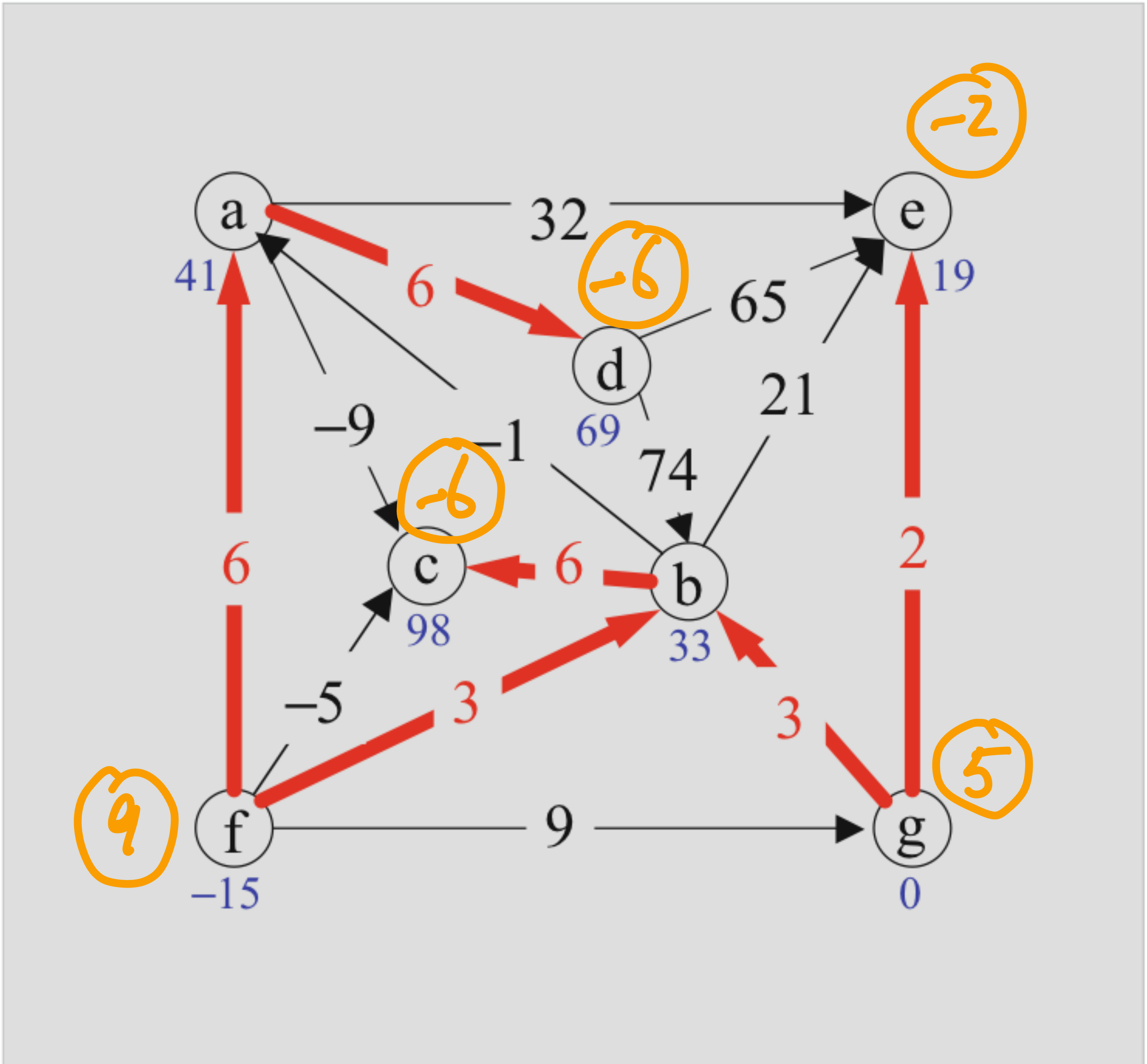
Spanning Trees and Bases



Spanning Trees and Bases

shown in Figure 14.1. They were obtained by starting at the "leaves" of the tree and working "inward." For instance, the flows could be solved for successively as follows:

leaf →

$$\begin{array}{lrcl}
\text{flow bal at d:} & x_{\text{ad}} = 6, & & \\
\text{flow bal at a:} & x_{\text{fa}} - x_{\text{ad}} = 0 & \Longrightarrow & x_{\text{fa}} = 6, \\
\text{flow bal at f:} & -x_{\text{fa}} - x_{\text{fb}} = -9 & \Longrightarrow & x_{\text{fb}} = 3, \\
\text{flow bal at c:} & x_{\text{bc}} = 6, & & \\
\text{flow bal at b:} & x_{\text{fb}} + x_{\text{gb}} - x_{\text{bc}} = 0 & \Longrightarrow & x_{\text{gb}} = 3, \\
\text{flow bal at e:} & x_{\text{ge}} = 2. & &
\end{array}$$

It is easy to see that this process always works. The reason is that every tree must have at least one leaf node, and deleting a leaf node together with the edge leading into it

Primal Flow

Spanning Trees and Bases

The above computation suggests that spanning trees are related to bases in the simplex method. Let us pursue this idea. Normally, a basis is an invertible square submatrix of the constraint matrix. But for incidence matrices, no such submatrix exists. To see why, note that if we sum together all the rows of A, we get a row vector of all zeros (since each column of A has exactly one $+1$ and one -1). Of course, every square submatrix of A has this same property and so is singular. In fact, we shall show in a moment that for a connected network, there is exactly one redundant equation (i.e., the rank of A is exactly $m - 1$).

Let us select some node, say, the last one, and delete the flow balance constraint associated with this node from the constraints defining the problem (since it is redundant anyway). Let us call this node the *root node*. Let $\tilde{A}$ denote the incidence matrix A without the row corresponding to the root node (i.e., the last row), and let $\tilde{b}$ denote the supply/demand vector with the last entry deleted. The most important property of network flow problems is summarized in the following theorem:

THEOREM 14.1. *A square submatrix of $\tilde{A}$ is a basis if and only if the arcs to which its columns correspond form a spanning tree.*

Spanning Trees and Bases

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(if : $B x_B = -\tilde{b}$; only if: Ex. 14.12)

Spanning Trees and Bases

THEOREM 14.1. *A square submatrix of $\tilde{A}$ is a basis if and only if the arcs to which its columns correspond form a spanning tree.*

basic variables

$$\tilde{A} = [B \quad N], \quad X = \begin{pmatrix} X_B \\ X_N \end{pmatrix}$$

$$[B \quad N]\begin{bmatrix} X_B \\ X_N \end{bmatrix} = -b$$

$$BX_B + NX_N = -b$$

$$X_B + (B^{-1}N)X_N = -B^{-1}b$$

$X_B \geq 0$, $X_N = 0$

T – spanning tree

$X_{ij} \neq 0 \quad (i,j) \in T$

$X_{ij} = 0 \quad (i,j) \notin T$

Spanning Trees and Bases

Dual Flow

$$y_j - y_i + z_{ij} = c_{ij} \qquad (i,j) \in \mathcal{A}$$

$$(z_{ij} \geq 0)$$

Dual Flow for Spanning Tree (T)

(Complementary Slackness $x_{ij} z_{ij} = 0$)

$(i,j) \in T$, $\quad x_{ij} \neq 0 \Rightarrow z_{ij} = 0$

$$y_j - y_i = c_{ij}$$

$(i,j) \in \mathcal{A} \setminus T$, $\quad z_{ij} = c_{ij} - (y_j - y_i)$

Spanning Trees and Bases

$y_j - y_i = c_{ij}$

For example, let node "g" be the root node in the spanning tree in Figure 14.6. Starting with it, we compute the dual variables as follows:

root node

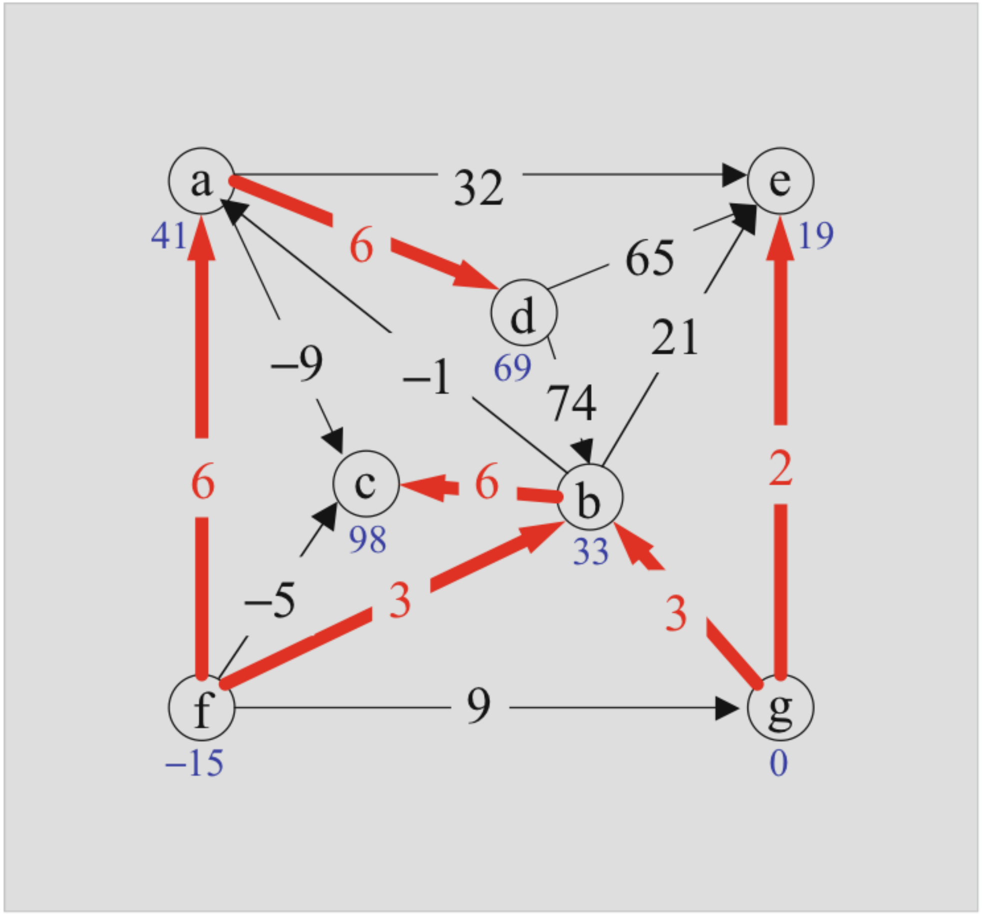
$$
\begin{array}{llll}
 & y_{\mathrm{g}} = 0, & & \\
\text{across arc (g,e):} & y_{\mathrm{e}} - y_{\mathrm{g}} = 19 & \Longrightarrow & y_{\mathrm{e}} = 19, \\
\text{across arc (g,b):} & y_{\mathrm{b}} - y_{\mathrm{g}} = 33 & \Longrightarrow & y_{\mathrm{b}} = 33, \\
\text{across arc (b,c):} & y_{\mathrm{c}} - y_{\mathrm{b}} = 65 & \Longrightarrow & y_{\mathrm{c}} = 98, \\
\text{across arc (f,b):} & y_{\mathrm{b}} - y_{\mathrm{f}} = 48 & \Longrightarrow & y_{\mathrm{f}} = -15, \\
\text{across arc (f,a):} & y_{\mathrm{a}} - y_{\mathrm{f}} = 56 & \Longrightarrow & y_{\mathrm{a}} = 41, \\
\text{across arc (a,d):} & y_{\mathrm{d}} - y_{\mathrm{a}} = 28 & \Longrightarrow & y_{\mathrm{d}} = 69.
\end{array}
$$

Now that we know the dual variables, the dual slacks for the arcs not in the spanning tree $\mathcal{T}$ can be computed using

$$z_{ij} = y_i + c_{ij} - y_j, \qquad (i,j) \notin \mathcal{T}$$

(which is just the dual feasibility condition solved for z_{ij}). These values are shown on

Spanning Trees and Bases



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Spanning Trees and Bases

From duality theory, we know that the current tree solution is optimal if all the flows are nonnegative and if all the dual slacks are nonnegative. The tree solution shown in Figure 14.6 satisfies the first condition but not the second. That is, it is primal feasible but not dual feasible. Hence, we can apply the primal simplex method to move from this solution to an optimal one. We take up this task in the next section.