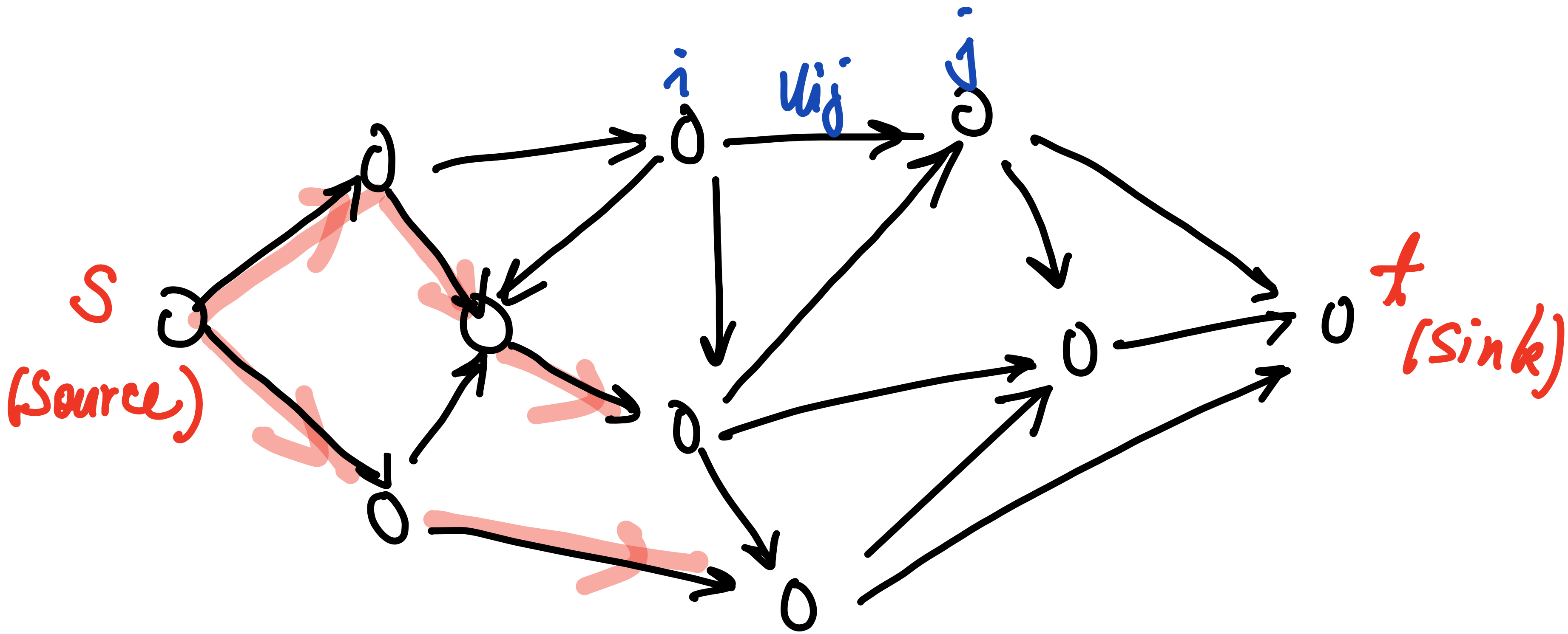
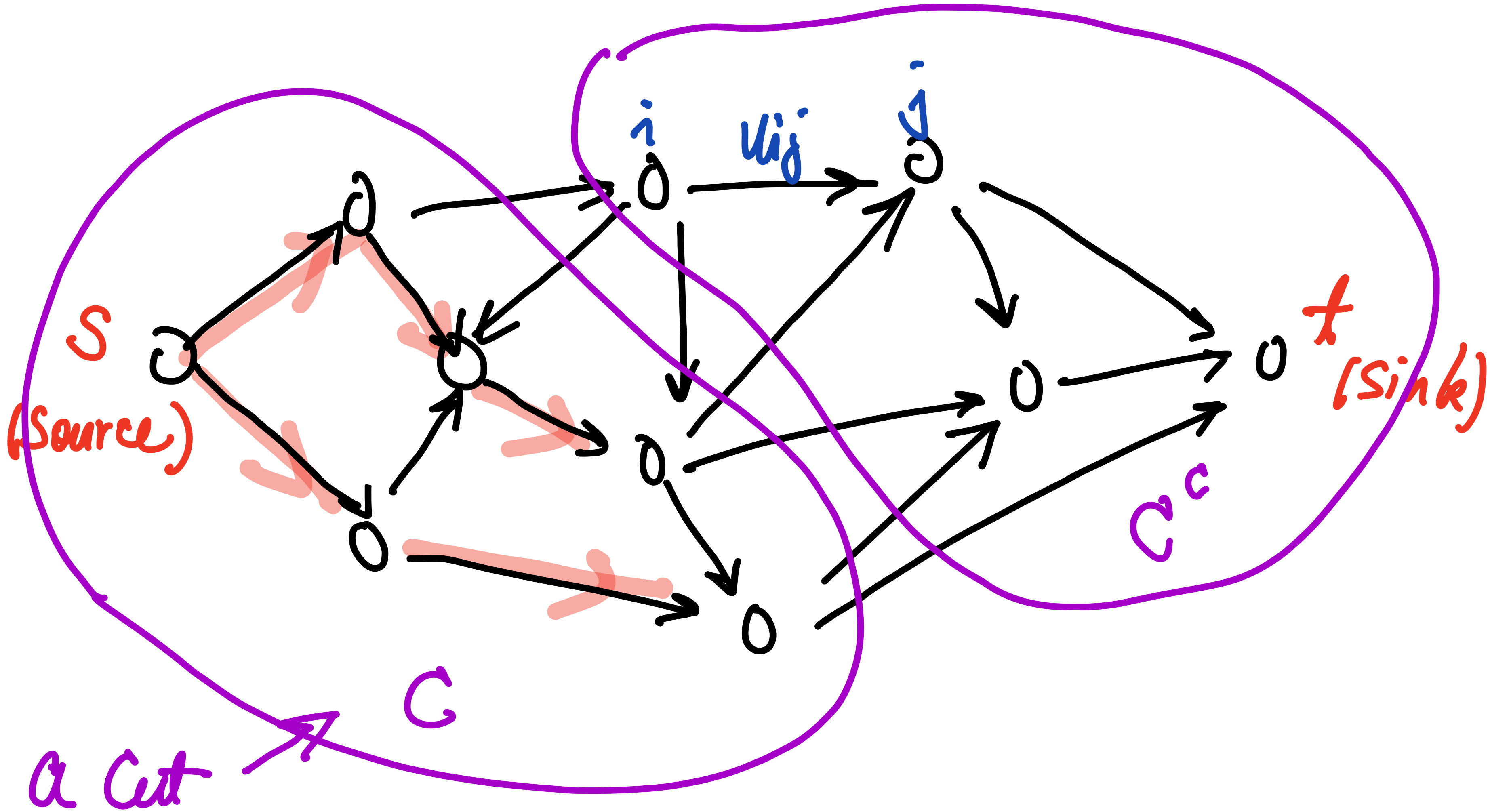
Max Flow Min Cut



What is the max. flow from s to t subject to

$$\boxed{0 \leq x_{ij} \leq u_{ij}} \quad ?$$

Max Flow Min Cut

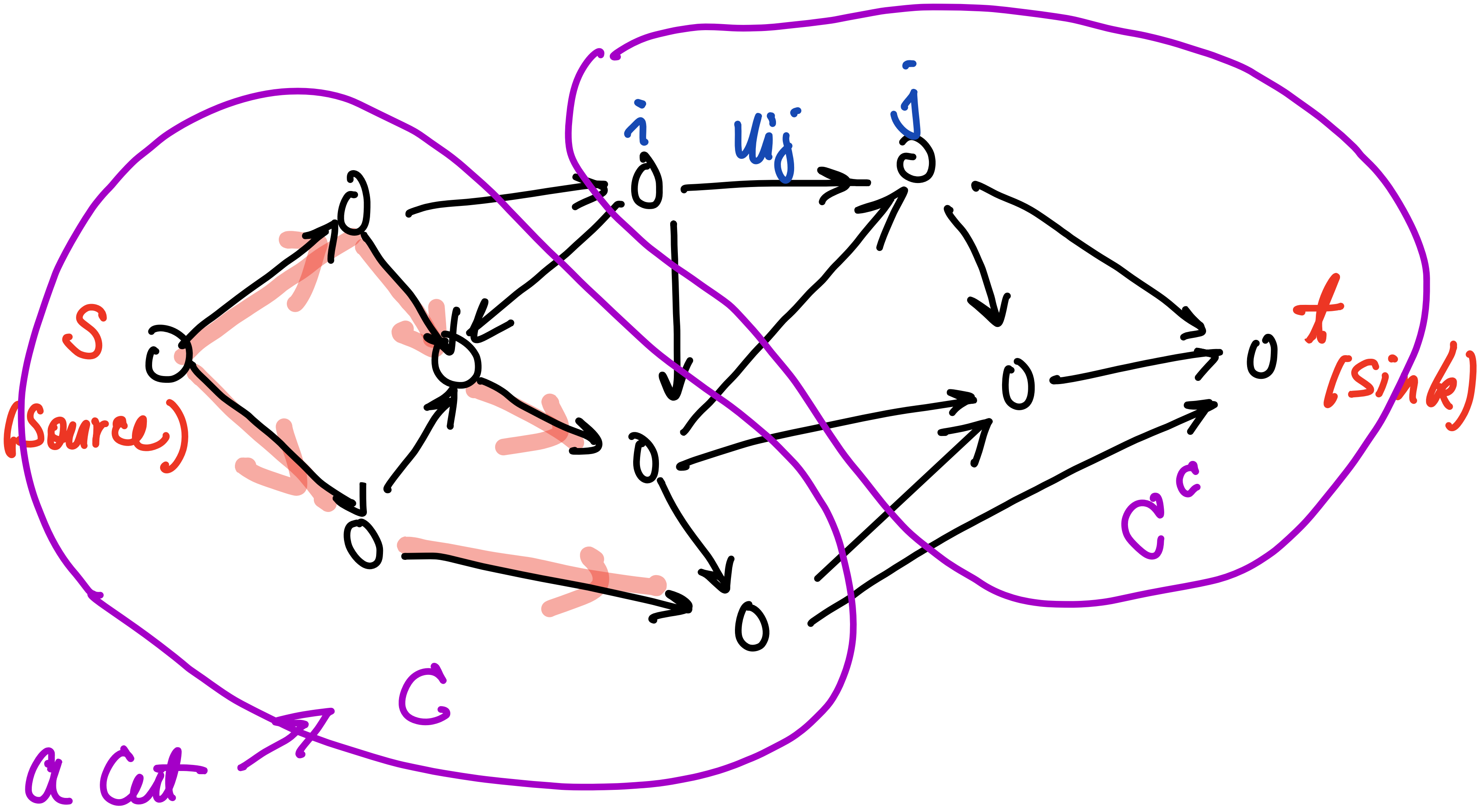


$$N = C \cup C^c, \quad s \in C \;\; (t \in C^c)$$

$$\boxed{K(C) = \sum_{i \in C,\, j \notin C} u_{ij}}$$

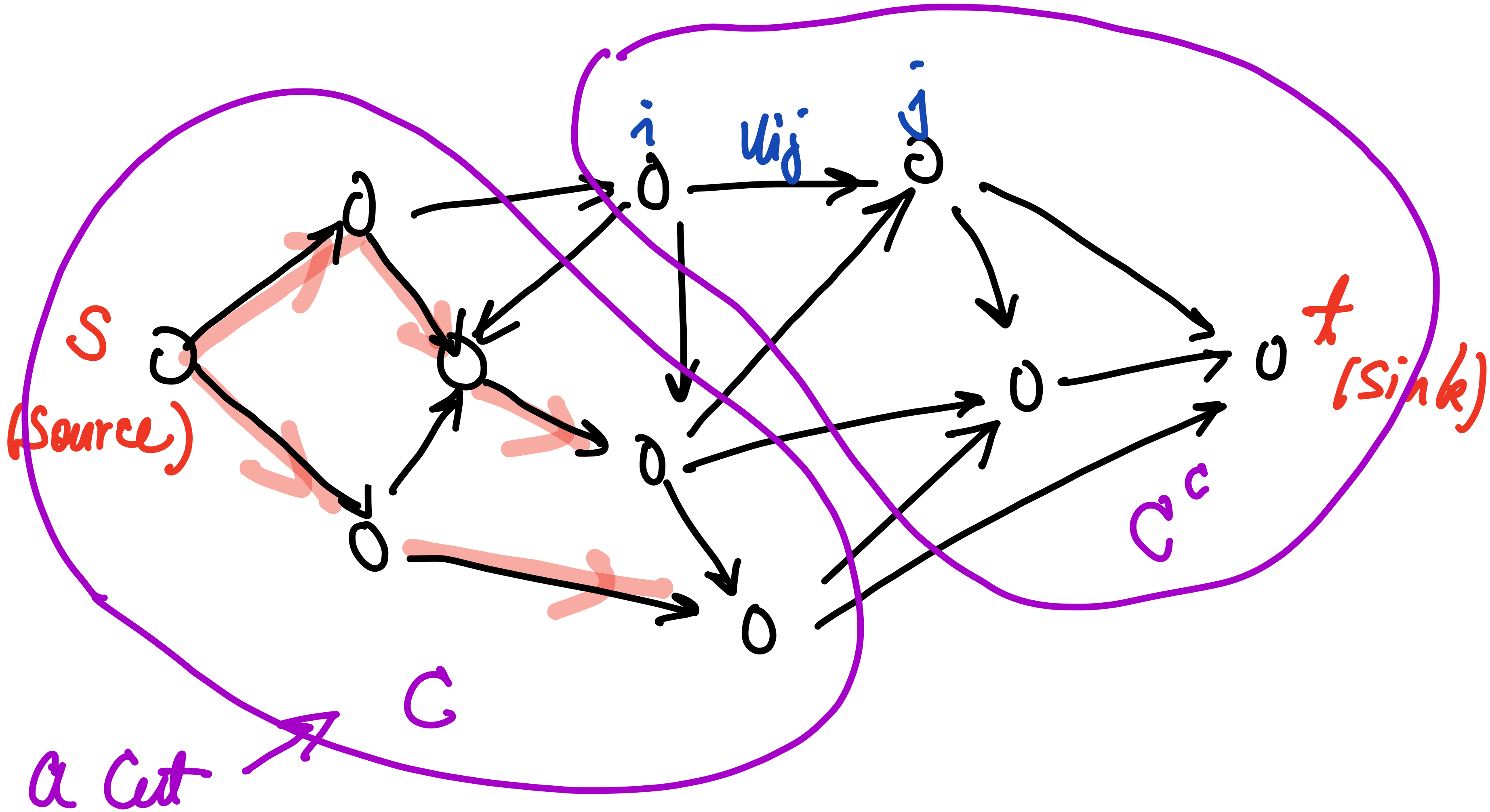
capacity of the cut C.

Max Flow Min Cut



(any) Flow(s → t) ≤ (any) $\kappa(C)$

Max Flow Min Cut



$$\cancel{\text{(any)}}_{\text{max}} \ \text{Flow}(s \to t) \ \underset{=}{\leq} \ \cancel{\text{(any)}}_{\text{min}} \ K(C)$$

[C] p.368

Table 22.1 The Travelers' Example: Seat Availability

From	To	Number of seats
San Francisco	Denver	5
San Francisco	Houston	6
Denver	Atlanta	4
Denver	Chicago	2
Houston	Atlanta	5
Atlanta	New York	7
Chicago	New York	4

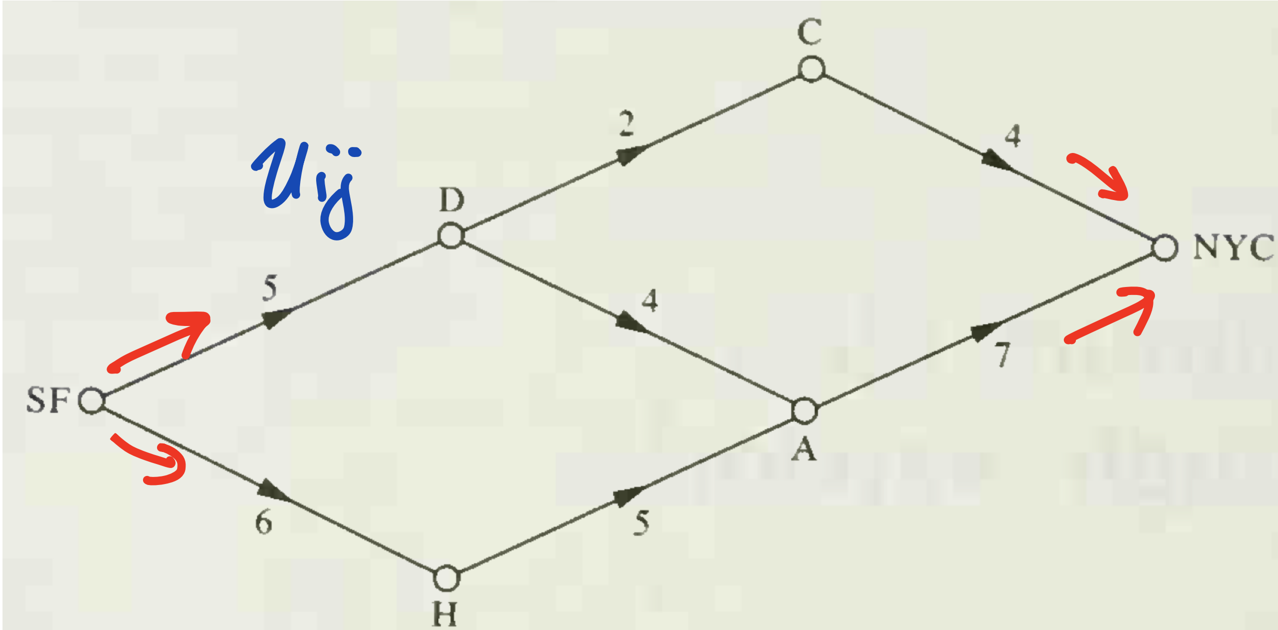


Figure 22.1

[C] p.368

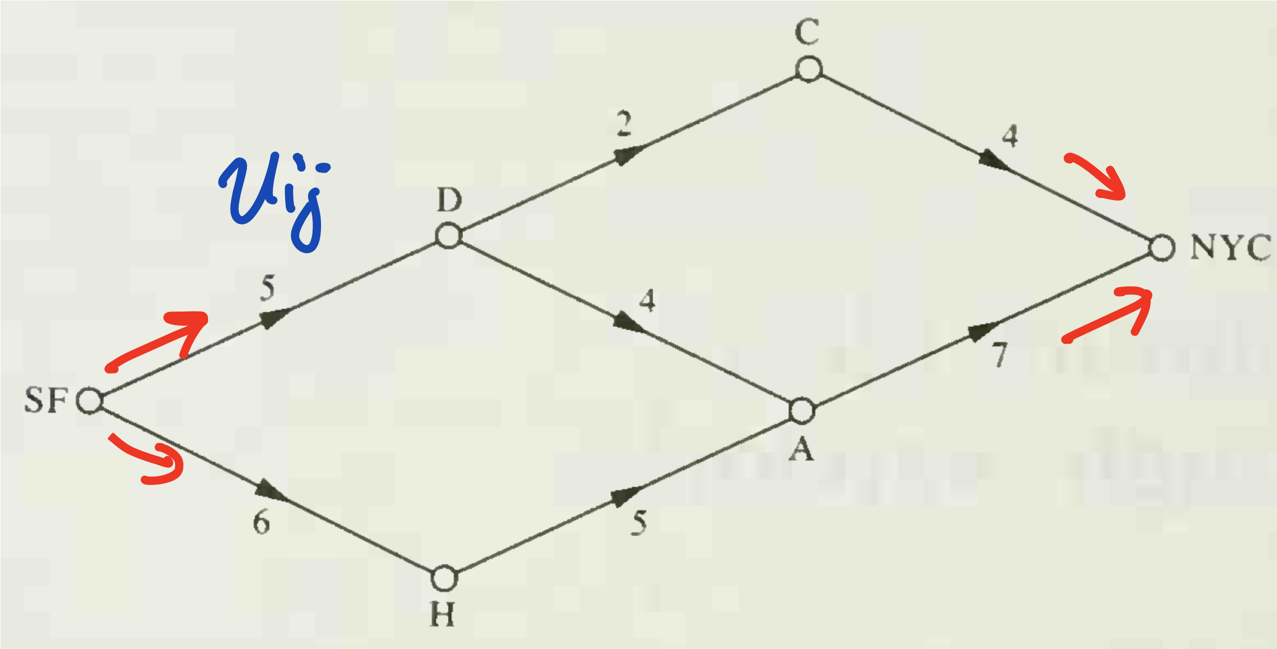


Figure 22.1

Flow = 8

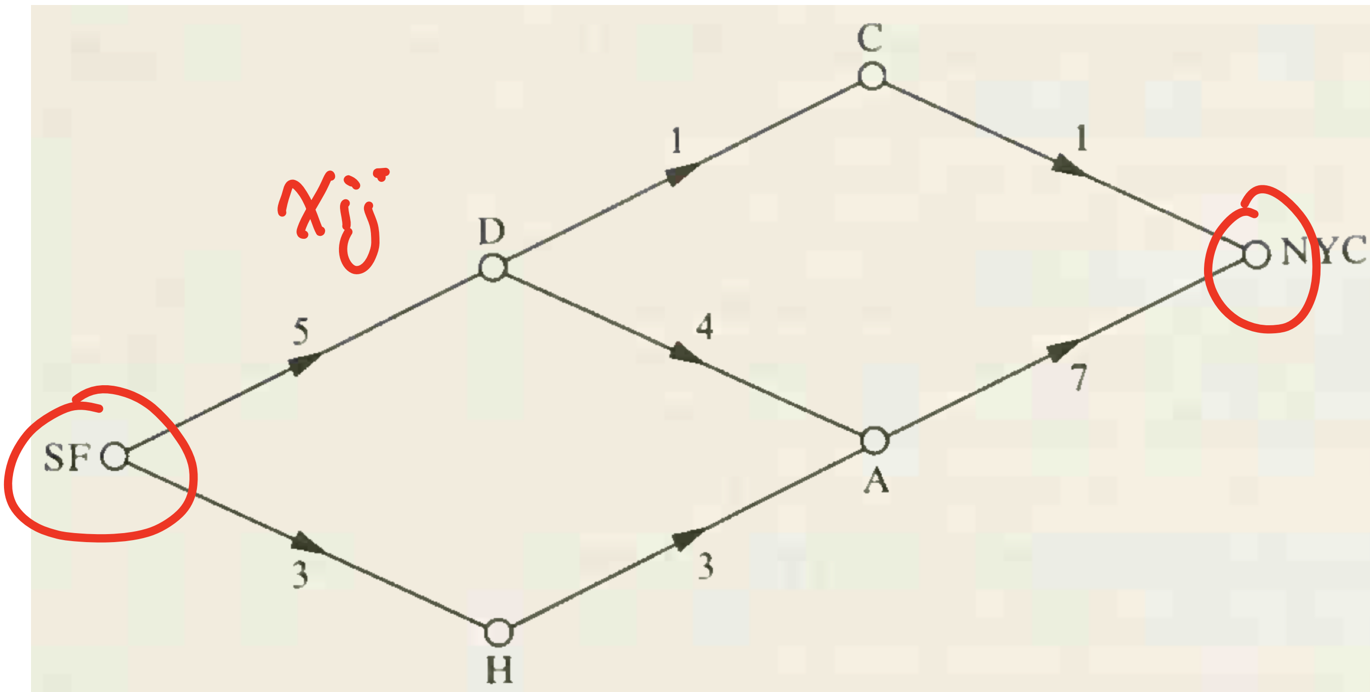


Figure 22.2

[C] p.368

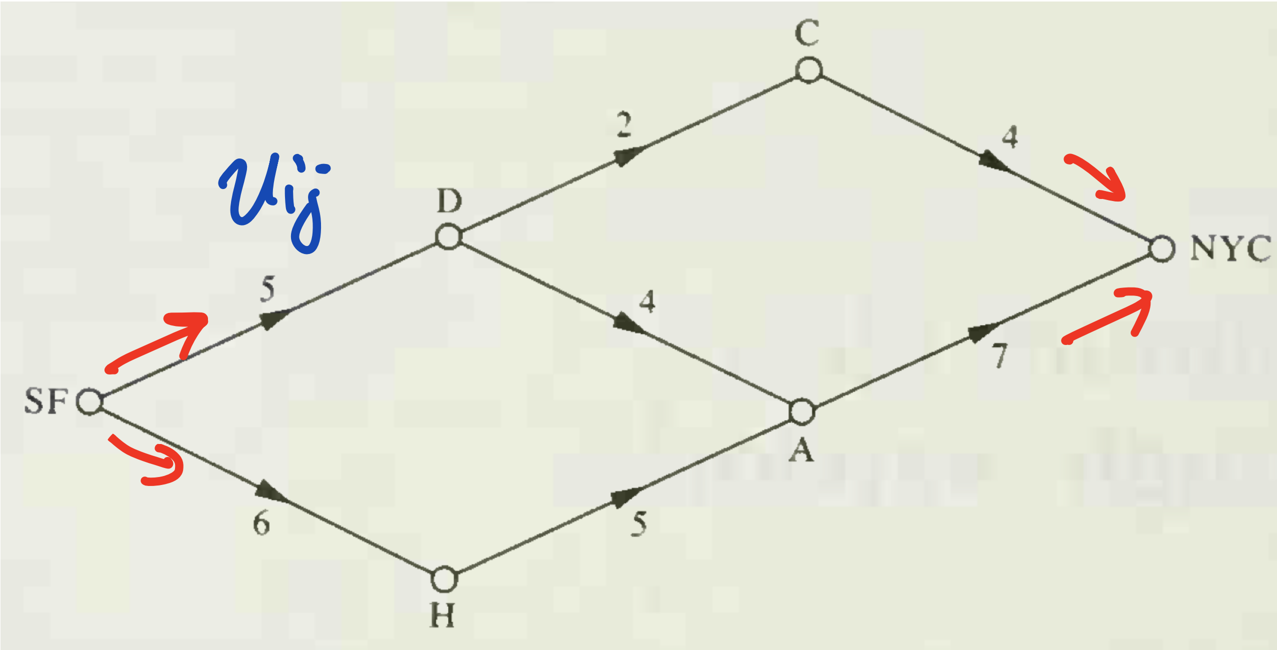


Figure 22.1

Flow = 9

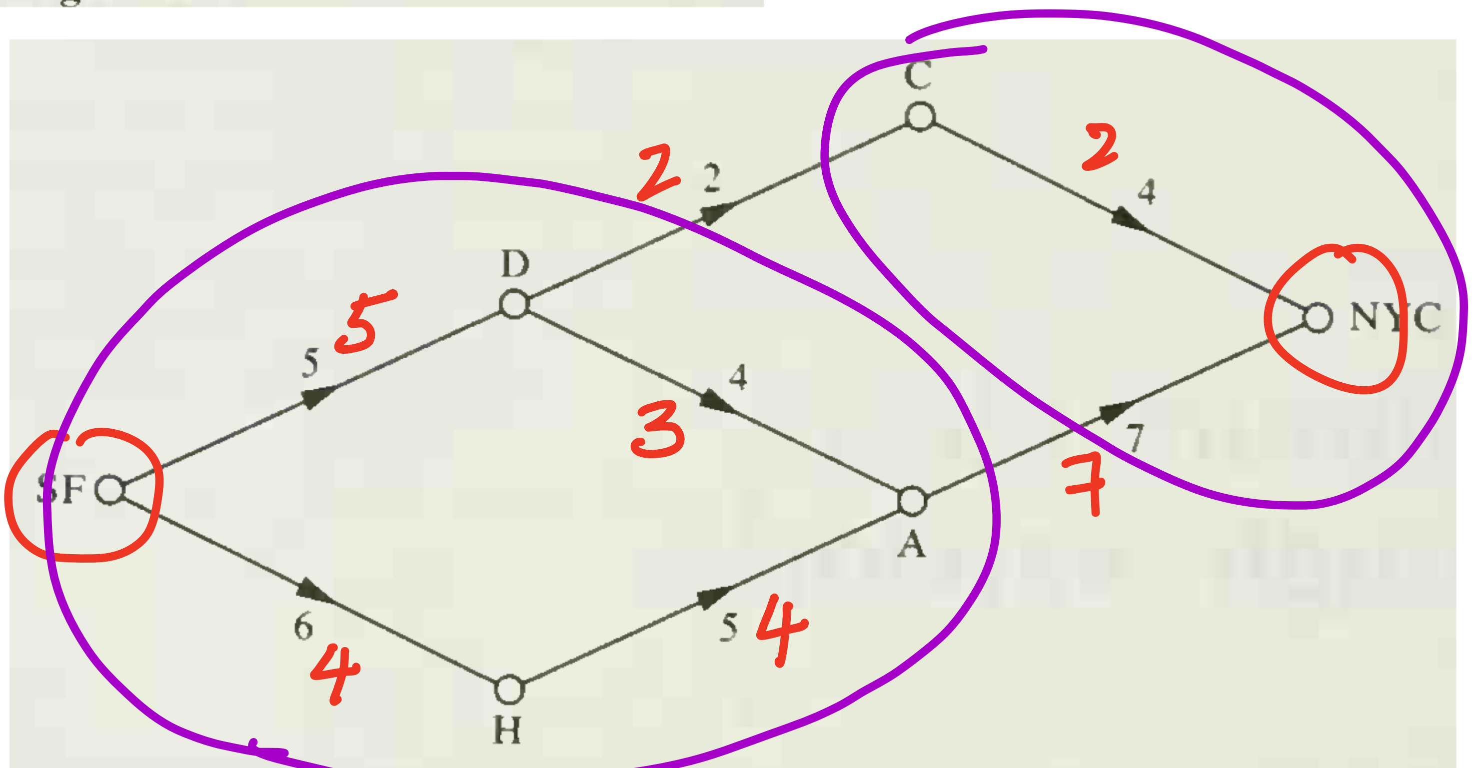


Figure 22.1

Proof of Max Flow = Min Cut

- Let $\{x_{ij}^*\}$ be a (optimal) max flow
- Let C^* be those nodes that some more flow can be pushed from s.
- $t \notin C^*$ (By default, $s \in C^*$)

Claim

$$\text{Flow}(C^* \text{ to } C^{*c}) = K(C^*)$$

$$\sum_{i \in C^*, j \notin C^*} x_{ij}^* = \sum_{i \in C^*, j \notin C^*} u_{ij}$$

[C] p.368

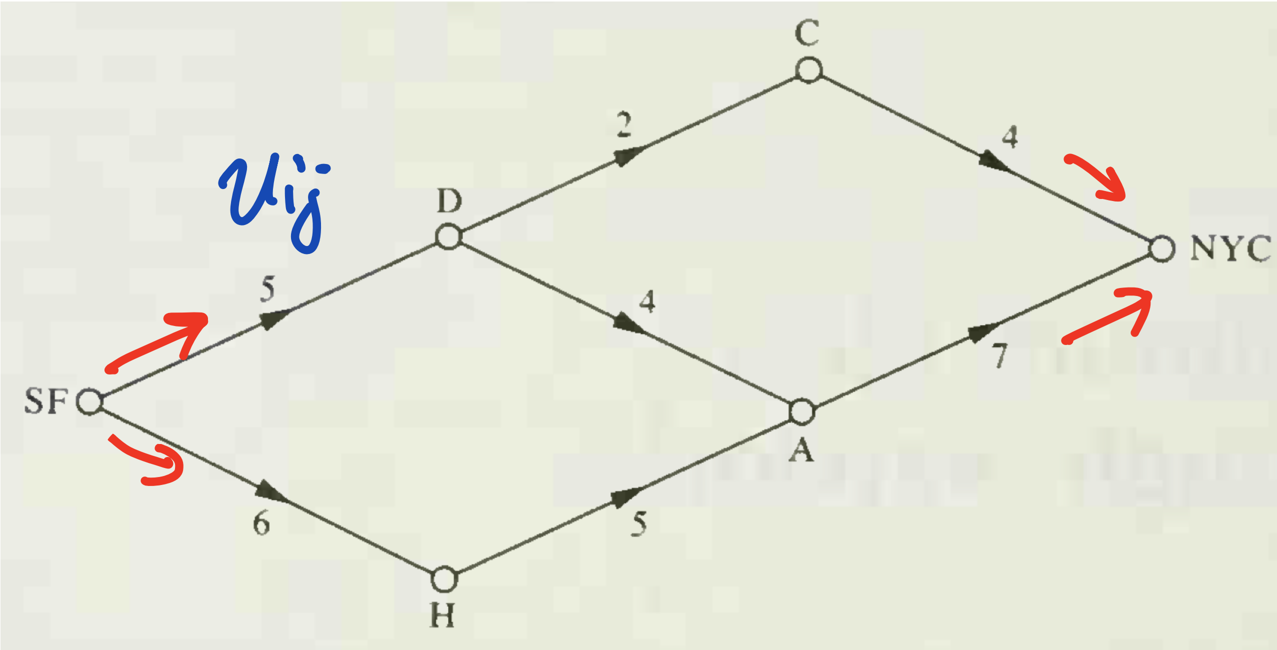


Figure 22.1

Flow = 9

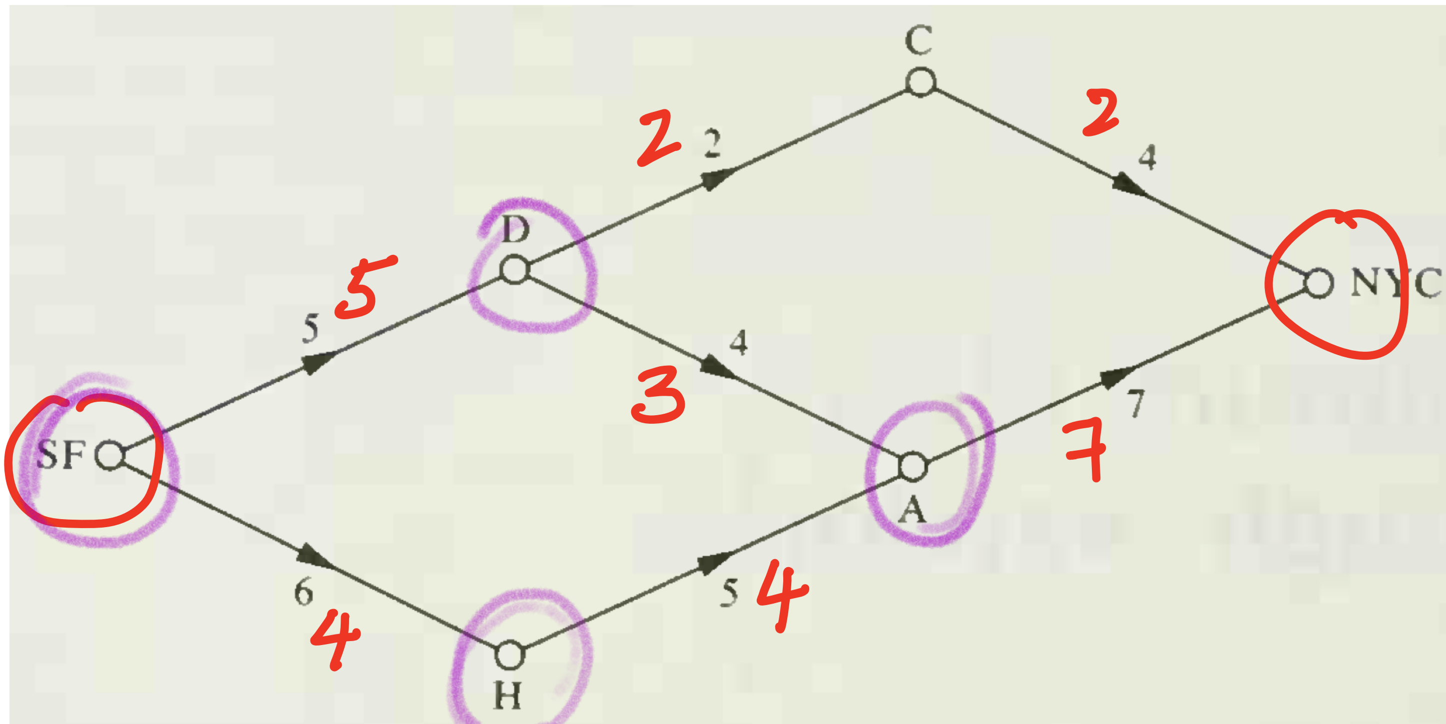
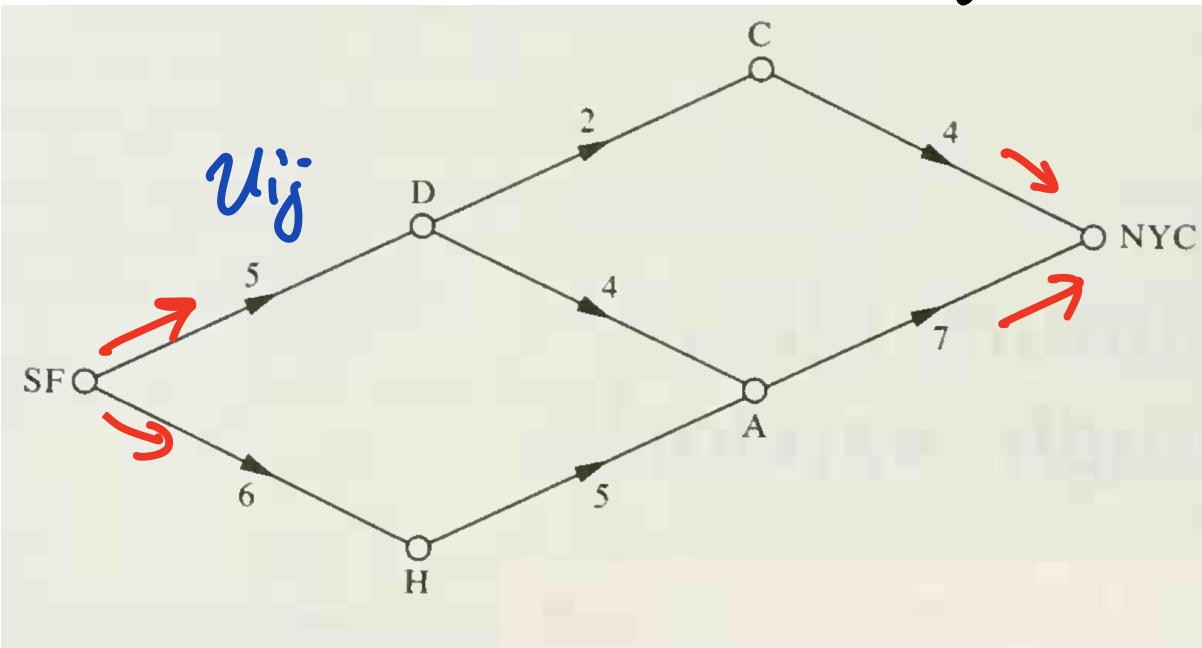


Figure 22.1

Ford-Fulkerson Algorithm (Augmented Path)



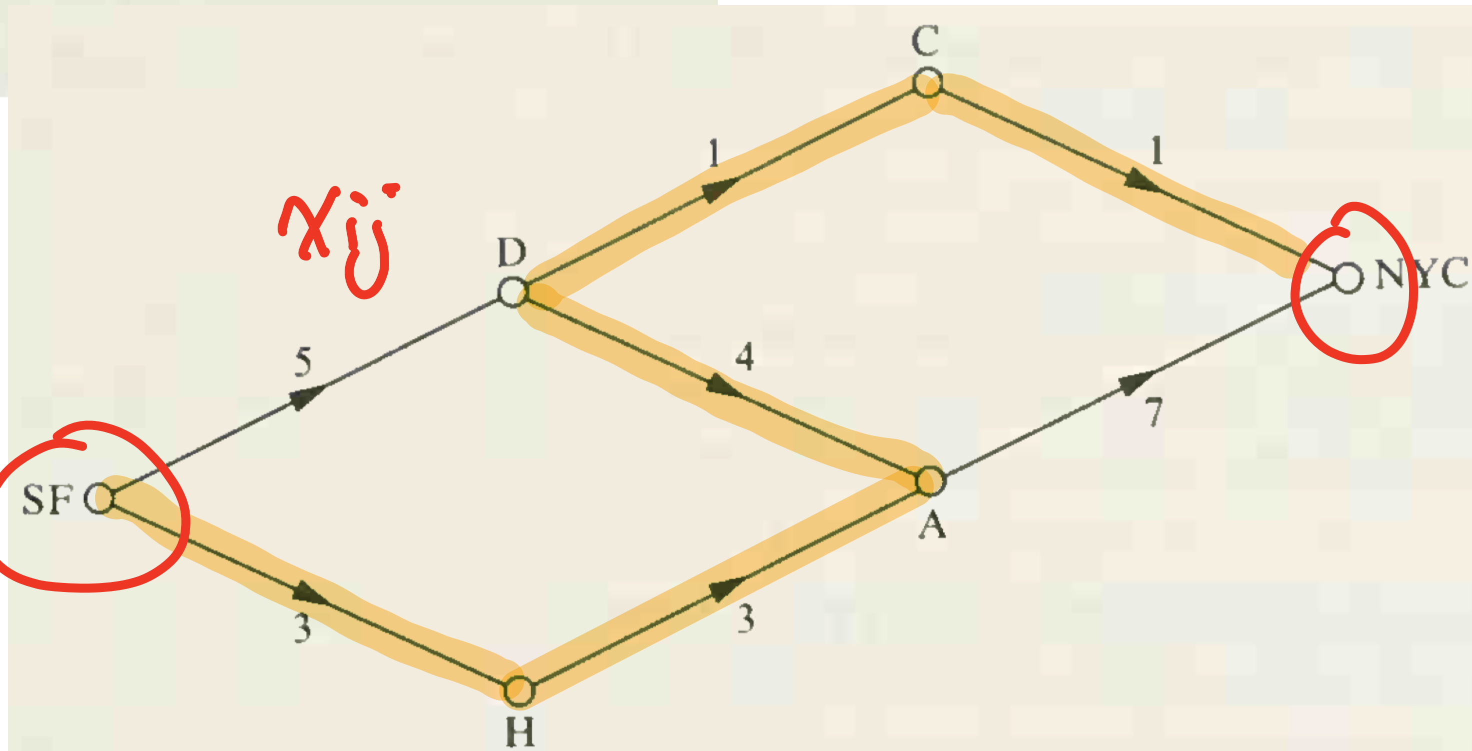
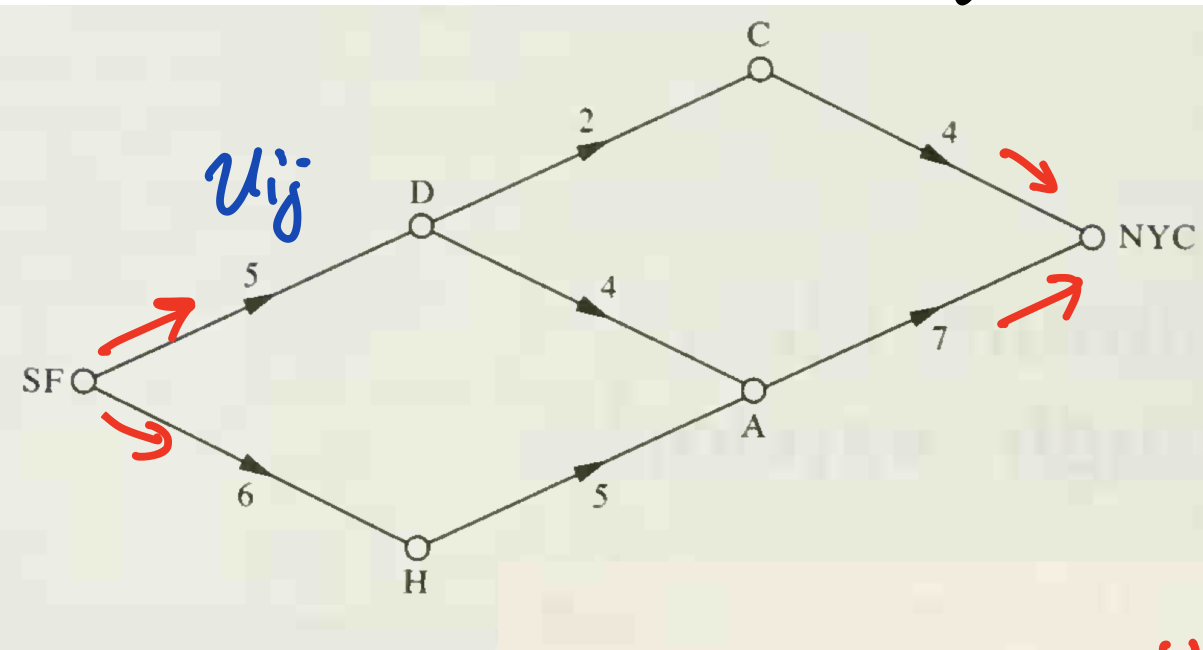


Figure 22.2

Ford-Fulkerson Algorithm (Augmented Path)

[C] p. 374



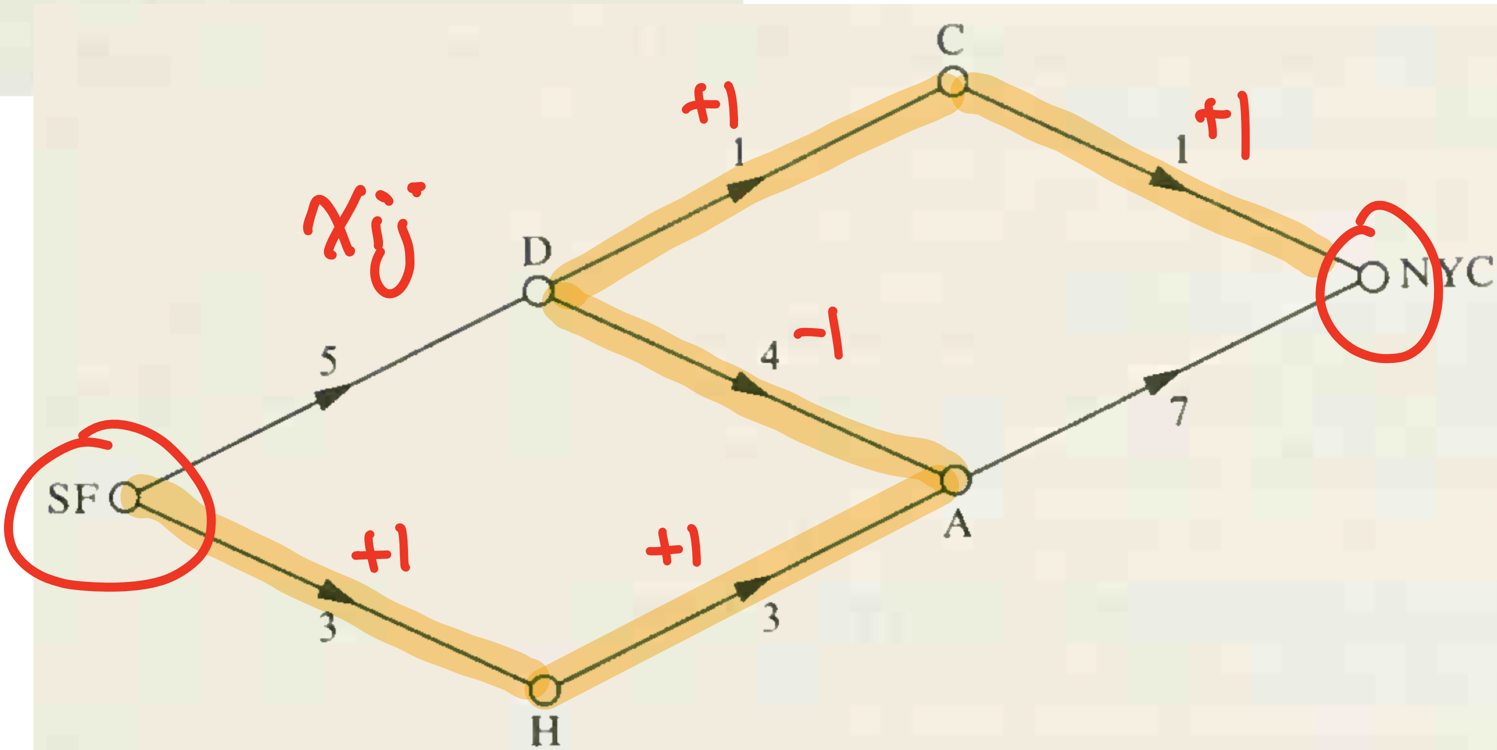


Figure 22.2

Ford- Fulkerson Algorithm (Augmented Path)

Implementations

Our description of the augmenting path method does not specify a way of searching for the arcs ij such that $i \in C$, $j \notin C$, $x_{ij} < u_{ij}$, and the arcs ji such that $j \notin C$, $i \in C$, $x_{ji} > 0$. Ford and Fulkerson did specify a way of doing that. In their terminology, nodes in C are called *labeled* and nodes outside C are called *unlabeled*; the labeled nodes are divided further into *scanned* and *unscanned*. Initially, the source s is labeled but unscanned and all the remaining nodes are unlabeled. Scanning a labeled node i means *examining all the arcs* ij *and, whenever such an arc satisfies* $x_{ij} < u_{ij}$, $j \notin C$, *setting* $j \in C$, $p(j) = ij$ and *examining all the arcs* ji *and, whenever such an arc satisfies* $x_{ji} > 0$, $j \notin C$, *setting* $j \in C$, $p(j) = ji$. The search may be described as in Box 22.1.

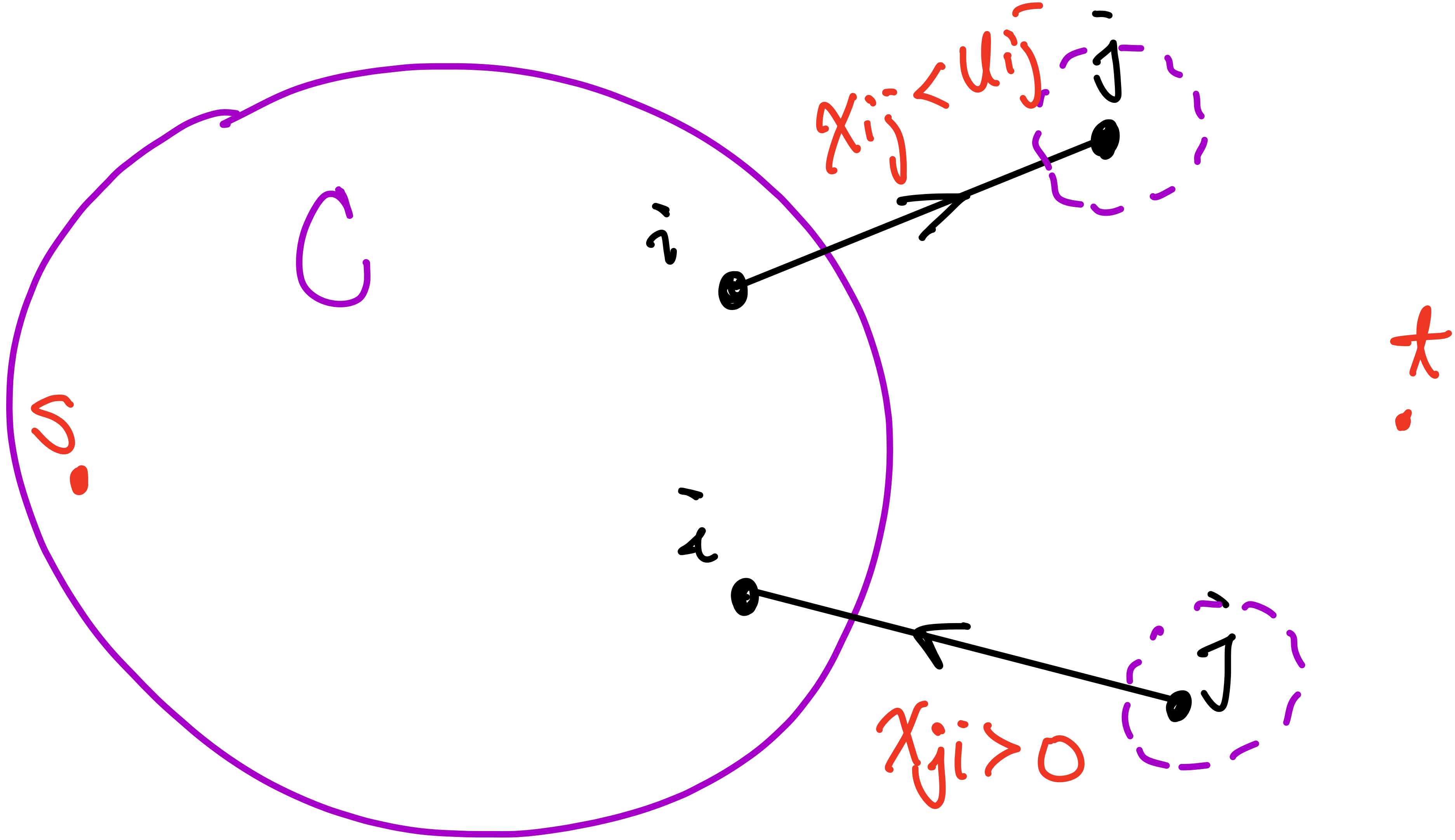
BOX 22.1 Search for an Augmenting Path

Step 0. Mark s as labeled unscanned; mark the remaining nodes as unlabeled.

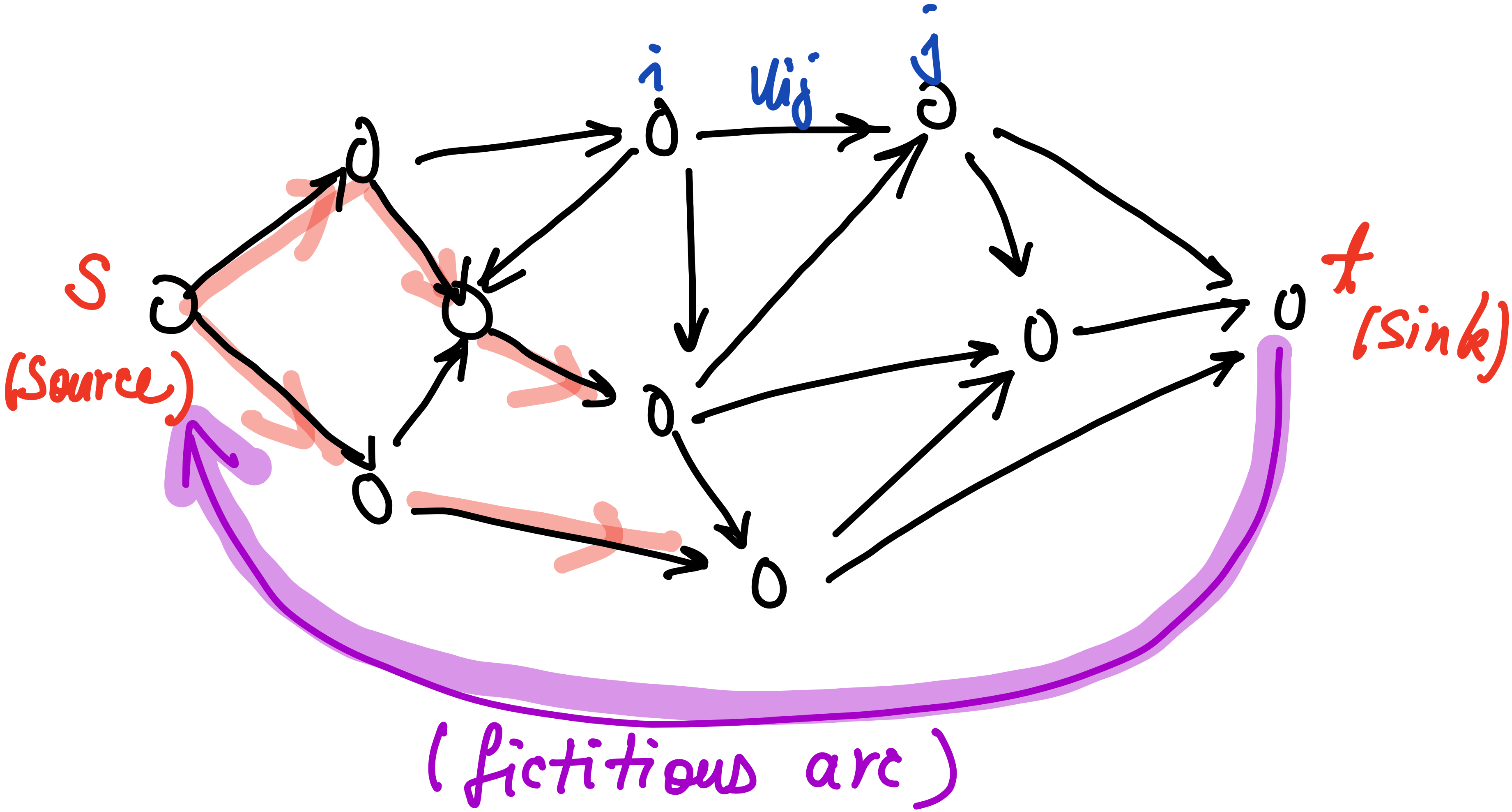
Step 1. If all the labeled nodes are scanned then stop [the set C of labeled nodes satisfies (22.7) and (22.8)]; otherwise, choose a labeled unscanned node i.

Step 2 Scan i. If t has become labeled then stop (an $\mathbf{x}$-augmenting path has been found); otherwise return to step 1.

Ford-Fulkerson Algorithm (Augmented Path)



Max Flow Min Cut (LP Formulation)



$x_{ts}, \quad c_{ts} = -1, \quad u_{ts} = +\infty$

Max Flow Min Cut (LP Formulation)

$$\min \quad -x_{ts}$$

s.t.

$$x_{ts} = \sum_j x_{sj}$$

$$\sum_i x_{it} = x_{ts}$$

$$\sum_i x_{ik} = \sum_j x_{kj} \quad k \neq s,t$$

$$0 \leq x_{ij} \leq u_{ij}$$

$$(u_{ts} = +\infty)$$

Proof of Max Flow = Min Cut (LP)

$0 \leqslant x_{ij} \leqslant u_{ij}$

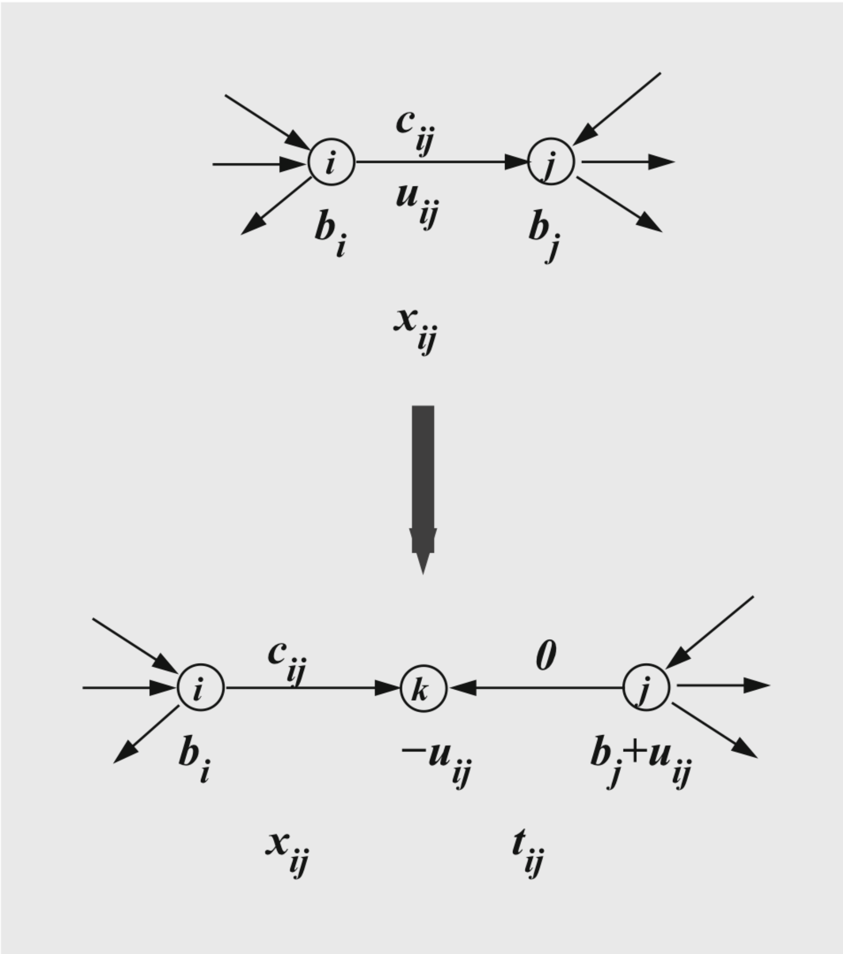


FIGURE 15.4. Adding a new node, k, to accommodate an arc (i, j) having an upper bound u_{ij} on its flow capacity.

$$x_{ij} = 0 \implies y_i + c_{ij} \geq y_j$$
$$x_{ij} = u_{ij} \implies y_i + c_{ij} \leq y_j$$
$$0 < x_{ij} < u_{ij} \implies y_i + c_{ij} = y_j.$$

$x_{ij} + t_{ij} = u_{ij}$

Proof of Max Flow = Min Cut (LP)

Let x^*_{ij}, $(i, j) \in \mathcal{A}$, denote the optimal values of the primal variables, and let y^*_i, $i \in \mathcal{N}$, denote the optimal values of the dual variables. Then the complementarity conditions (15.6) imply that

$$\boxed{\begin{aligned} x^*_{ij} &= 0 \;\text{ whenever }\; y^*_i + c_{ij} > y^*_j \quad &(15.9)\\ x^*_{ij} &= u_{ij} \;\text{ whenever }\; y^*_i + c_{ij} < y^*_j. \quad &(15.10)\end{aligned}}$$

In particular,

$$y^*_t - 1 \geq y^*_s$$

(since $u_{ts} = \infty$). Put $C^* = \{k : y^*_k \leq y^*_s\}$. Clearly, C^* is a cut.

Consider an arc having its tail in C^* and its head in the complement of C^*. It follows from the definition of C^* that $y^*_i \leq y^*_s < y^*_j$. Since c_{ij} is zero, we see from (15.10) that $x^*_{ij} = u_{ij}$.

Now consider an original arc having its tail in the complement of C^* and its head in C^* (i.e., bridging the two sets in the opposite direction). It follows then that $y^*_j \leq y^*_s < y^*_i$. Hence, we see from (15.9) that $x^*_{ij} = 0$.