MA 421: Linear Programming and Optimization Techniques
Fall 2024, Midterm
Instructor: Yip

- This test booklet has FOUR QUESTIONS, totaling 100 points for the whole test. You have 75 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, closed note, with no electronic devices**.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.
- **As a rule of thumb, you should give explicit and useful answers.** No points will be given for just writing down some generically true statements. In other words, your answers should try to make use of all the information given in the question.
- **As a rule of thumb, you should only use those methods that have been covered in class.** If you use some other methods "for the sake of convenience", at our discretion, we might not give you any credit. You have the right to contest. In that event, **you will be asked to explain your answer using only what has been covered in class up to the point of time of this exam.**

Name: Answer Key (Major:)

Question	Score
1.(20 pts)	
2.(20 pts)	
3.(30 pts)	
4.(30 pts)	
Total (100 pts)	

Formula sheet.

Matrix form of simplex method.

Given the following linear program problem in its standard form:

$$\begin{array}{ll} \text{maximize} & c^T X \\ \text{subject to} & AX \le b; \\ & X \ge 0. \end{array}$$

During simplex iterations, the above can be transformed into the following matrix form:

$$\begin{aligned} \zeta &= c_{\mathcal{B}}^T (B^{-1} b) - \big((B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}}\big)^T X_{\mathcal{N}} \\ X_{\mathcal{B}} &= B^{-1} b - \big(B^{-1}N\big) X_{\mathcal{N}} \end{aligned}$$

with the following dual form:

$$\begin{aligned} -\xi &= -c_{\mathcal{B}}^T (B^{-1} b) - \big(B^{-1} b\big)^T Z_{\mathcal{B}} \\ Z_{\mathcal{N}} &= \big((B^{-1}N)^T c_{\mathcal{B}} - c_{\mathcal{N}}\big) + \big(B^{-1}N\big)^T Z_{\mathcal{B}} \end{aligned}$$

where in the above

1. $\mathcal{B}$ and $\mathcal{N}$ are the basic and non-basic variable indices;
2. $c_{\mathcal{B}}$ and $c_{\mathcal{N}}$ are the coefficients in the objective function corresponding to the basic and non-basic variables;
3. B and N are matrices formed by the collecting the columns from the augmented matrix $[A \;\; I]$ corresponding to the basic and non-basic variables.
4. $X_{\mathcal{B}}$ and $X_{\mathcal{N}}$ are the basic and non-basic primal variables, and $Z_{\mathcal{B}}$ and $Z_{\mathcal{N}}$ are the basic and non-basic dual variables.

1. Find the duals of the following two linear programming problems:

   (a)

$$
\begin{array}{rl}
\text{maximize} & c^T X \\
\text{subject to} & AX \le p; \\
 & BX = q.
\end{array}
$$

   (In the above, $X \in \mathbb{R}^n$ is the unknown vector, and c, p, q, A, B are some given (column) vectors and matrices with appropriate dimensions.)

   (b)

$$
\begin{array}{rlcl}
\text{maximize} & x_1 - 2x_2 & & \\
\text{subject to} & -(x_1 + 2x_2 - x_3 + x_4) & \overset{\le}{\ge} & 0 \\
 & 4x_1 + 3x_2 + 4x_3 - 2x_4 & \le & 3 \\
 & -x_1 - 2x_2 + x_3 + x_4 & = & 1
\end{array}
$$

(a)

$$
\begin{array}{ll}
y^T(AX \le p) & y \ge 0 \\
z^T(BX = q) & \\
\hline
c^T X \le (y^T A + z^T B)X \le y^T p + z^T q &
\end{array}
$$

$$
\boxed{
\begin{array}{ll}
\min\limits_{y,z} & y^T p + z^T q \\
\text{s.t.} & y^T A + z^T B = c^T \\
 & y \ge 0
\end{array}
}
$$

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(b)

$$+\begin{cases} y_1(-x_1-2x_2+x_3-x_4) \le 0 & y_1 \ge 0 \\ y_2(4x_1+3x_2+4x_3-2x_4) \le 3y_2 & y_2 \ge 0 \\ y_3(-x_1-2x_2+x_3+x_4) = y_3 \end{cases}$$

$$x_1-2x_2 \le (-y_1+4y_2-y_3)x_1 + (-2y_1+3y_2-2y_3)x_2 + (y_1+4y_2+y_3)x_3 + (-y_1-2y_2+y_3)x_4 \le 3y_2+y_3$$

$$\boxed{\begin{aligned} \min\ & 3y_2+y_3 \\ \text{s.t.}\ & -y_1+4y_2-y_3 = 1 \\ & -2y_1+3y_2-2y_3 = -2 \\ & y_1+4y_2+y_3 = 0 \\ & -y_1-2y_2+y_3 = 0 \\ & y_1, y_2 \ge 0 \end{aligned}}$$

2. Consider the following linear programming problem:

$$
\begin{array}{lrll}
\text{maximize} & 5x_1 + 4x_2 + 3x_3 & & \\
\text{subject to} & 2x_1 + 3x_2 + x_3 & \leq 5 & = \\
& 4x_1 + x_2 + 2x_3 & \leq 11 & < \\
& 3x_1 + 4x_2 + 2x_3 & \leq 8 & = \\
& x_1, x_2, x_3 & \geq 0 &
\end{array}
$$

Someone claims that the optimal point is $(x_1^* = 2, x_2^* = 0, x_3^* = 1)$. Prove or disprove the claim by *not solving the problem explicitly.*

$x_1^* = 2 > 0 \Longrightarrow z_1 = 0$

$x_3^* = 1 > 0 \Longrightarrow z_3 = 0$

$4(2) + (0) + 2(1) < 11 \Longrightarrow$

$$
\begin{cases}
2y_1 + 4y_2 + 3y_3 = 5 \\
y_1 + 2y_2 + 2y_3 = 3 \\
y_2 = 0
\end{cases}
$$

$\Longrightarrow \quad y_3 = 1, \ y_1 = 1$

Check also $3y_1 + y_2 + 4y_3 > 4$

Hence $(y_1 = 1, y_2 = 0, y_3 = 1)$ found by complementary slackness satisfies the dual constraint.

Hence $(x_1 = 2, x_2 = 0, x_3 = 1)$ is opt.

Check also (but not necessary):

$5(2) + 4(0) + 3(1) = 13 = 5(1) + 11(0) + 8(1)$

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3. Solve the following linear programming problems.

$$(a) \quad \text{maximize} \quad x_1 + 2x_2 - 3x_3 - 4x_4$$
$$\text{subject to} \quad 0 \le x_1 \le 5,\; 0 \le x_2 \le 3,\; 0 \le x_3 \le 2,\; 0 \le x_4 \le 6$$

$$(b) \quad \text{maximize} \quad x_1 + 2x_2 - 3x_3 - 4x_4$$
$$\text{subject to} \quad 4x_1 + 10x_2 - x_3 - x_4 \le 5,\; x_1, x_2, x_3, x_4 \ge 0$$

$$(c) \quad \text{maximize} \quad x_1 + 2x_2 - 3x_3 - 4x_4$$
$$\text{subject to} \quad x_1 + x_2 - 5x_3 - 7x_4 \le 3,\; x_1, x_2, x_3, x_4 \ge 0$$

For each of the above problem, find the optimal objective functional value and also the values of the variables at the optimal point. Maybe the dual problems might help.

As always, you should give sufficient (but not necessarily long) explanations of how to obtain your answers.

(a)

$$(a) \quad \text{maximize} \quad \underline{x_1 + 2x_2} - \underline{3x_3 - 4x_4}$$
$$\text{subject to} \quad 0 \le x_1 \le 5,\; 0 \le x_2 \le 3,\; 0 \le x_3 \le 2,\; 0 \le x_4 \le 6$$

try to make x_1, x_2 big, x_3, x_4 small

$$\boxed{\text{Set } x_1 = 5,\ x_2 = 3,\ x_3 = 0,\ x_4 = 0 \Rightarrow \zeta = 11}$$

(b) (dual) min $5y$

s.t.
$$\left.\begin{array}{l} 4y \geq 1 \\ 10y \geq 2 \\ -y \geq -3 \\ -y \geq -4 \\ y \geq 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} y \geq \frac{1}{4} \\ y \geq \frac{1}{5} \\ 3 \geq y \\ 4 \geq y \\ y \geq 0 \end{array}\right.$$

$$\Rightarrow \underline{\tfrac{1}{4} \le y \le 3}$$

$\min 5y = 5 \times \frac{1}{4} = \frac{5}{4}$

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$\frac{1}{4} > \frac{1}{5} \Rightarrow x_2 = 0$

$\frac{1}{4} < 3 \Rightarrow x_3 = 0$

$\frac{1}{4} < 4 \Rightarrow x_4 = 0$

$\frac{1}{4} > 0 \Rightarrow w = 0$

i.e. $4x_1 + 10x_2 - x_3 - x_4 = 5$

$\Rightarrow x_1 = \frac{5}{4}$

$$\boxed{\text{Opt. Solution pt } \left(\tfrac{5}{4}, 0, 0, 0\right), \quad \zeta = \tfrac{5}{4}}$$

(c) (Dual) $\min 3y$

s.t.
$$\left.\begin{aligned} y &\geq 1 \\ y &\geq 2 \\ -5y &\geq -3 \\ -7y &\geq -4 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} y &\geq 1 \\ y &\geq 2 \\ y &\leq \tfrac{3}{5} \\ y &\leq \tfrac{4}{7} \end{aligned}\right.$$

empty set

Hence dual problem is not feasible.

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However, the primal problem is clearly feasible. Hence the primal problem is unbounded

4. Consider the following linear programming problem:

$$\begin{array}{lrl} \text{maximize} & x_1 + 2x_2 & \\ \text{subject to} & x_1 + x_2 & \leq 5 \\ & x_1 & \leq 4 \\ & x_2 & \leq 3 \\ & x_1, x_2 & \geq 0 \end{array}$$

(a) Solve the above problem and give also the dictionary at the optimal point.

(b) Suppose the constraint $x_1 + x_2 \leq 5$ is changed to $x_1 + x_2 \leq p$. Find the range of p such that the dictionary you have found in (a) remains optimal.

(c) Suppose the objective function $x_1 + 2x_2$ is changed to $x_1 + qx_2$. Find the range of q such that the dictionary you have found in (a) remains optimal.

(a) 1 max $\zeta = x_1 + 2x_2$ → x_2 enters and w_3 leaves

$w_1 = 5 - x_1 - x_2$

$w_2 = 4 - x_1$

$w_3 = 3 \quad - x_2$

$\Longrightarrow$ $x_2 = 3 - w_3$

$w_1 = 5 - x_1 - 3 + w_3 = 2 - x_1 + w_3$

$\zeta = x_1 + 2(3 - w_3) = 6 + x_1 - 2w_3$

2 max $\zeta = 6 + x_1 - 2w_3$ → x_1 enters, w_1 leaves

$w_1 = 2 - x_1 + w_3$

$w_2 = 4 - x_1$

$x_2 = 3 \quad - w_3$

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$$x_1 = 2 - w_1 + w_3$$

$$w_2 = 4 - 2 + w_1 - w_3$$

$$= 2 + w_1 - w_3$$

$$\zeta = 6 + 2 - w_1 + w_3 - 2w_3$$

$$= 8 - w_1 - 2w_3$$

3

$$\boxed{\begin{aligned} \max \quad \zeta &= 8 - w_1 - 2w_3 \\ x_1 &= 2 - w_1 + w_3 \\ w_2 &= 2 + w_1 - w_3 \\ x_2 &= 3 \qquad\quad - w_3 \end{aligned}}$$

$\leftarrow$ optimal

$\zeta^* = 8$, $x_1^* = 2$, $x_2^* = 3$, $w_2^* = 2$, $w_1^* = w_3^* = 0$

(b) Need to ensure $X_B^* = B^{-1} b_{new} \geq 0$

$$[A \quad I] = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$B = \{1, 4, 2\}$, $N = \{3, 5\}$

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$$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B^{-1}: \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

B^{-1}

$$B^{-1} b_{new} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} p-3 \\ -p+7 \\ 3 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{3 \leq p \leq 7}$$

ⓒ Need to ensure $Z_B^* \geq 0$

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$$Z_B^* = (B^{-1}N)^T C_B - C_N$$

$$B^{-1}N = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 1 \end{pmatrix}$$

in fact, can be read off from the opt. dictionary

$$C = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad C_B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad C_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2 \to q$

$$Z_{B\,new}^* = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ q \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1+q \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \boxed{q \geq 1}$$

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