[V] # 10.5

The proof of Thm 10.3 makes use of Thm 3.4.
But Thm 3.4 requires LP in standard form:
$Ax \le b$ while (10.2) is not in standard form.

So we need to transform (10.2) into standard form.

$$(10.2) \quad \begin{cases} Ax = b \\ e^T x = 1 \end{cases}, \quad (x \geq 0), \quad A^{m \times n}, \quad e, x \in \mathbb{R}^m$$

$$\iff \underbrace{\begin{bmatrix} A \\ e^T \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} x \end{bmatrix} = \underbrace{\begin{bmatrix} b \\ 1 \end{bmatrix}}_{\tilde{b}} \in \mathbb{R}^{m+1}$$

i.e. $$\tilde{A} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \tilde{b} \qquad \tilde{A}^{(m+1) \times n}$$

$$\underbrace{\begin{bmatrix} \tilde{u}_1 & \tilde{u}_2 & \cdots & \tilde{u}_n \end{bmatrix}}_{\text{cols of } \tilde{A}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \tilde{b}$$

Let $\operatorname{Rank}(\tilde{A}) = r \leq m+1 < n$

(if $n \leq m+1$, nothing to proof)

Without loss of generality, assume $\{\tilde{u}_1, \tilde{u}_2, \cdots \tilde{u}_r\}$ is linearly independent.

$$\Big[\underbrace{\tilde{u}_1 \ \tilde{u}_2 \ \cdots \ \tilde{u}_r}_{\tilde{B}} \ \underbrace{\tilde{u}_{r+1} \ \cdots \ \tilde{u}_n}_{\tilde{C}}\Big] \begin{bmatrix} x_1 \\ \vdots \\ x_r \\ x_{r+1} \\ \vdots \\ x_n \end{bmatrix} = \tilde{b}$$

$$\Longleftrightarrow \quad \underbrace{\tilde{B}}_{(m+1)\times r} \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix} + \underbrace{\tilde{C}}_{(m+1)\times(n-r)} \begin{bmatrix} x_{r+1} \\ \vdots \\ x_n \end{bmatrix} = \tilde{b}$$

$$\underbrace{\tilde{B}^T \tilde{B}}_{r \times r} \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix} + \underbrace{\tilde{B}^T \tilde{C}}_{r\times(n-r)} \begin{bmatrix} x_{r+1} \\ \vdots \\ x_n \end{bmatrix} = \underbrace{\tilde{B}^T \tilde{b}}_{r \times 1}$$

Note: as $\tilde{B}$ has linearly independent cols, $\tilde{B}^T \tilde{B}$ is invertible, i.e. $(\tilde{B}^T \tilde{B})^{-1}$ exists

$$\Rightarrow \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix}}_{\tilde{\tilde{W}}} + \underbrace{(\tilde{B}^T\tilde{B})^{-1}\tilde{B}^T\tilde{C}}_{\tilde{\tilde{A}}} \underbrace{\begin{bmatrix} x_{r+1} \\ \vdots \\ x_n \end{bmatrix}}_{\tilde{\tilde{X}}} = \underbrace{(\tilde{B}^T\tilde{B})^{-1}\tilde{b}}_{\tilde{\tilde{b}}}$$

i.e. $\underbrace{\tilde{\tilde{W}}}_{\geq 0} + \tilde{\tilde{A}}\tilde{\tilde{X}} = \tilde{\tilde{b}}$

i.e. $\tilde{\tilde{A}}\tilde{\tilde{X}} \leq \tilde{\tilde{b}}, \quad \tilde{\tilde{X}} \geq 0$

in standard form, with $r \leq m+1$ basic vars.

Hence Thm 3.4 can be used.

[C]

16.10 Derive the following theorems (with the vector inequality $\mathbf{v} > \mathbf{w}$ meaning, as usual, $v_k > w_k$ for all k) from the result of problem 16.9.

(i) P. Gordan (1873): The system $\mathbf{Ax} < \mathbf{0}$ is unsolvable if and only if the system $\mathbf{yA} = \mathbf{0}, \mathbf{y} \geq \mathbf{0}, \mathbf{y} \neq \mathbf{0}$ is solvable.

Prove: If there is a y s.t. $yA=0$, $y \geq 0$, $y \neq 0$.

Then $AX < 0$ is not solvable.

pf Suppose $AX<0$ is solvable, i.e. such an X exists.

$AX < 0 \Rightarrow y\cancel{A}X < y0 \quad (y \neq 0)$

$\Rightarrow 0 < 0$ Contradiction!

#2

$$\boxed{\begin{array}{lll} (P) & \min & t_1+t_2+t_3+t_4 \\ (0,0) \Rightarrow & & -t_1 \leq -b \leq t_1 \\ (2,1) \Rightarrow & & -t_2 \leq 1-2a-b \leq t_2 \\ (4,2) \Rightarrow & & -t_3 \leq 2-4a-b \leq t_3 \\ (1,p) \Rightarrow & & -t_4 \leq p-a-b \leq t_4 \end{array}}$$

$y = ax+b$

$a=\frac{1}{2}, \quad b=0$

$(0,0)$

$(2,1)$

$(4,2)$

$(1,p)$

Proposed solution : $a=\frac{1}{2}, \quad b=0$

$$\Rightarrow \quad \begin{array}{lll} -t_1 \leq 0 \leq t_1 & \Rightarrow & t_1 = 0 \\ -t_2 \leq 0 \leq t_2 & \Rightarrow & t_2 = 0 \\ -t_3 \leq 0 \leq t_3 & \Rightarrow & t_3 = 0 \\ -t_4 \leq p-\frac{1}{2} \leq t_4 & \Rightarrow & t_4 = \left|p-\frac{1}{2}\right| \end{array}$$

min value

$$\boxed{\min \; t_1+t_2+t_3+t_4 = \left|p-\frac{1}{2}\right|}$$

$$\boxed{\text{Proposed solution: } a=\frac{1}{2},\; b=0,\; t_1=t_2=t_3=0,\; t_4=\left|p-\frac{1}{2}\right|}$$

Find (D)

$$-t_1 \le -b \le t_1$$
$$-t_2 \le 1-2a-b \le t_2$$
$$-t_3 \le 2-4a-b \le t_3$$
$$-t_4 \le p-a-b \le t_4$$

Standard form:

$$\Longrightarrow \begin{array}{ll}
u_1 (& b + t_1 \ge 0) \\
u_2 (& -b + t_1 \ge 0) \\
u_3 (& 2a + b + t_2 \ge 1) \\
u_4 (& -2a - b + t_2 \ge -1) \\
u_5 (& 4a + b + t_3 \ge 2) \\
u_6 (& -4a - b + t_3 \ge -2) \\
u_7 (& a + b + t_4 \ge p) \\
u_8 (& -a - b + t_4 \ge -p)
\end{array}$$

$u_i \ge 0$
$i = 1, 2, \cdots, 8$

$$t_1 + t_2 + t_3 + t_4$$
$$\ge \big(2(u_3 - u_4) + 4(u_5 - u_6) + (u_7 - u_8) \big) a$$
$$+ \big((u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + (u_7 - u_8) \big) b$$
$$+ (u_1 + u_2) t_1 + (u_3 + u_4) t_2 + (u_5 + u_6) t_3 + (u_7 + u_8) t_4$$
$$\ge (u_3 - u_4) + 2(u_5 - u_6) + p(u_7 - u_8)$$

$$\boxed{\begin{aligned} &\max\ (u_3-u_4)+2(u_5-u_6)+p(u_7-u_8)\\ &\text{s.t. } 2(u_3-u_4)+4(u_5-u_6)+(u_7-u_8)=0\\ &(u_1-u_2)+(u_3-u_4)+(u_5-u_6)+(u_7-u_8)=0\\ &u_1+u_2=1,\ u_3+u_4=1,\ u_5+u_6=1,\ u_7+u_8=1\\ &u_1,u_2,\dots,u_8\geq 0 \end{aligned}}$$

(D)

Let

$\underline{V_1=u_1-u_2}$, $\underline{V_2=u_3-u_4}$, $\underline{V_3=u_5-u_6}$, $\underline{V_4=u_7-u_8}$

(D): $\max\ \xi = V_2+2V_3+pV_4$

s.t. $2V_2+4V_3+V_4=0$ (1)

$V_1+V_2+V_3+V_4=0$ (2)

$V_1=1-2u_2,\quad 0\leq u_2\leq 1 \Leftarrow (u_1\geq 0)$

$V_2=1-2u_4,\quad 0\leq u_4\leq 1 \Leftarrow (u_3\geq 0)$

$V_3=1-2u_6,\quad 0\leq u_6\leq 1 \Leftarrow (u_5\geq 0)$

$V_4=1-2u_8,\quad 0\leq u_8\leq 1 \Leftarrow (u_7\geq 0)$

① $\Rightarrow \quad V_2 + 2V_3 = -\frac{V_4}{2}$

$$\Rightarrow \xi = -\frac{V_4}{2} + p V_4 = \left(p - \frac{1}{2}\right) V_4$$

$$\Rightarrow \boxed{\max \left(p - \frac{1}{2}\right) V_4}$$

①, ② $\Rightarrow$ solve for V_2, V_3

$$V_2 + 2V_3 = -\frac{V_4}{2}$$
$$V_2 + V_3 = -V_1 - V_4$$

$$\Rightarrow \boxed{\begin{aligned} V_3 &= V_1 + \frac{V_4}{2} \\ V_2 &= -2V_1 - \frac{3V_4}{2} \end{aligned}}$$

$\leftarrow$ eliminate V_2, V_3

$$\left.\begin{aligned} &V_1 = 1 - 2u_2 && \underline{0 \leq u_2 \leq 1} \\ \checkmark\ &V_2 = 1 - 2u_4 && 0 \leq u_4 \leq 1 \\ \checkmark\ &V_3 = 1 - 2u_6 && 0 \leq u_6 \leq 1 \\ &V_4 = 1 - 2u_8, && \underline{0 \leq u_8 \leq 1} \end{aligned}\right\}$$

$\leftarrow$ eliminate V_2, V_3

$$V_2 = 1 - 2u_4 \Rightarrow \quad u_4 = \frac{1}{2} - \frac{1}{2} V_2$$

$$= \frac{1}{2} - \frac{1}{2}\left(-2V_1 - \frac{3V_4}{2}\right)$$

$$u_4 = \frac{1}{2} + V_1 + \frac{3}{4} V_4$$

$$\boxed{\begin{aligned} u_4 &= \frac{1}{2} + 1 - 2u_2 + \frac{3}{4}(1 - 2u_8) \\ &= \frac{9}{4} - 2u_2 - \frac{3}{2} u_8 \end{aligned}}$$

$$0 \leq u_4 \leq 1$$

$$0 \leq \frac{9}{4} - 2u_2 - \frac{3}{2} u_8 \leq 1$$

$$\boxed{\frac{5}{8} \leq u_2 + \frac{3}{4} u_8 \leq \frac{9}{8}} \qquad 0 \leq u_4 \leq 1$$

$$V_3 = 1 - 2u_6 \Rightarrow \quad u_6 = \frac{1}{2} - \frac{1}{2} V_3$$

$$= \frac{1}{2} - \frac{1}{2}\left(V_1 + \frac{V_4}{2}\right)$$

$$u_6 = \frac{1}{2} - \frac{1}{2} V_1 - \frac{V_4}{4}$$

$$\boxed{\begin{aligned} u_6 &= \frac{1}{2} - \frac{1}{2}(1 - 2u_2) - \frac{1}{4}(1 - 2u_8) \\ &= -\frac{1}{4} + u_2 + \frac{u_8}{2} \end{aligned}}$$

$$0 \leq u_6 \leq 1$$

$$0 \leq -\frac{1}{4} + u_2 + \frac{u_8}{2} \leq 1$$

$$\boxed{\frac{1}{4} \leq u_2 + \frac{u_8}{2} \leq \frac{5}{4}} \qquad 0 \leq u_6 \leq 1$$

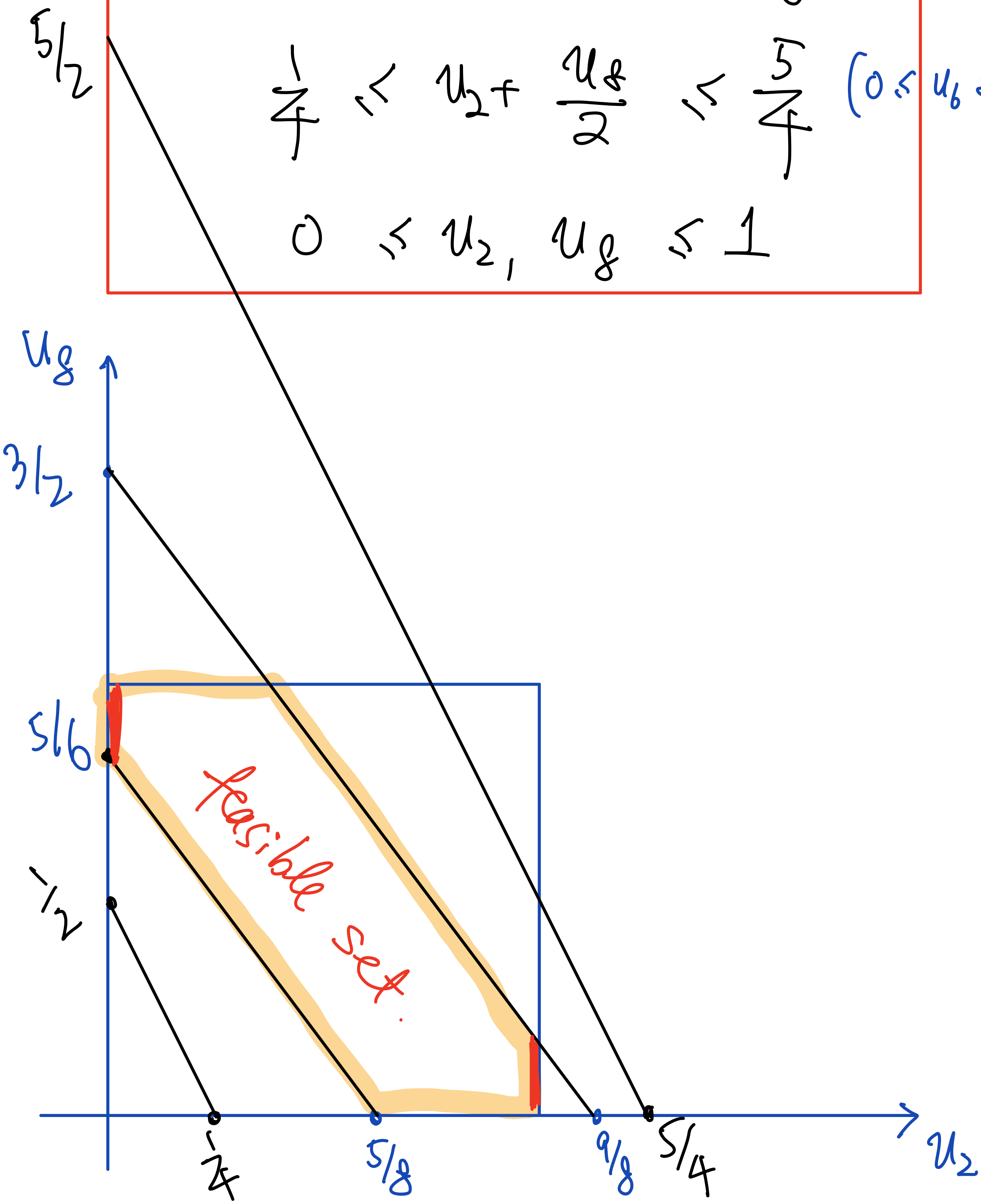
D

$$\max \quad \left(p - \tfrac{1}{2}\right)\left(1 - 2u_8\right)$$

$$\text{s.t.} \quad \frac{5}{8} \le u_2 + \frac{3}{4}u_8 \le \frac{9}{8} \qquad (0 \le u_4 \le 1)$$

$$\frac{1}{4} \le u_2 + \frac{u_8}{2} \le \frac{5}{4} \qquad (0 \le u_6 \le 1)$$

$$0 \le u_2,\ u_8 \le 1$$



If $p > \frac{1}{2}$, $\max\left(p-\frac{1}{2}\right)(1-2u_g)$

$= p-\frac{1}{2}$, (at $u_g = 0$)

If $p < \frac{1}{2}$ $\max\left(p-\frac{1}{2}\right)(1-2u_g)$

$= -\left(p-\frac{1}{2}\right)$, (at $u_g = 1$)

$$\boxed{\text{Hence } \max\left(p-\frac{1}{2}\right)(1-2u_g) = \left|p-\frac{1}{2}\right|}$$

#4 (a)

$$P_1 = \left\{ \begin{array}{l} -x_1 \le 0 \\ \frac{x_1}{2} - x_2 \le -5 \\ -x_1 - x_2 \le -10 \end{array} \right\} = \underbrace{\begin{bmatrix} -1 & 0 \\ \frac{1}{2} & -1 \\ -1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \underbrace{\begin{bmatrix} 0 \\ -5 \\ -10 \end{bmatrix}}_{b}$$

$$P_2 = \left\{ \begin{array}{l} -x_2 \le 0 \\ -x_1 + x_2 \le 0 \\ 3x_1 + x_2 \le 30 \end{array} \right\} = \underbrace{\begin{bmatrix} 0 & -1 \\ -1 & 1 \\ 3 & 1 \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \underbrace{\begin{bmatrix} 0 \\ 0 \\ 30 \end{bmatrix}}_{\tilde{b}}$$

$$P_1 \cap P_2 = \left\{ \begin{array}{l} Ax \le b \\ \tilde{A}x \le \tilde{b} \end{array} \right. = \boxed{\begin{bmatrix} A \\ \tilde{A} \end{bmatrix}} x \le \boxed{\begin{bmatrix} b \\ \tilde{b} \end{bmatrix}}$$

Apply FL to this system.

ie.

$$\boxed{\begin{array}{ll} \min & \begin{pmatrix} b^T & \tilde{b}^T \end{pmatrix} \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} \\ \text{s.t.} & \begin{pmatrix} A^T & \tilde{A}^T \end{pmatrix} \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} = 0 \\ & y, \tilde{y} \ge 0 \end{array}}$$

Using simplex to find $y, \tilde{y}$ s.t.

$$\boxed{\begin{gathered} \begin{pmatrix} A^T & \tilde{A}^T \end{pmatrix} \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} = 0, \quad y, \tilde{y} \geq 0 \\ \text{and} \quad \begin{pmatrix} b^T & \tilde{b}^T \end{pmatrix} \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} < 0 \end{gathered}}$$

$$\underline{A^T y + \tilde{A}^T \tilde{y} = 0} \qquad b^T y + \tilde{b}^T \tilde{y} < 0$$

Then

$$H_1 = \{x : (y^T A)x \leq y^T b\} \qquad (\leftarrow = b^T y)$$

$$H_2 = \{x : \underbrace{(\tilde{y}^T \tilde{A})x \leq \tilde{y}^T b}_{\parallel}\} \qquad \leftarrow = \tilde{b}^T \tilde{y}$$

$$(-y^T A)x \leq \tilde{y}^T b$$

$$-\tilde{y}^T b \leq (y^T A)x \quad \Leftarrow \text{same as [V] p. 164}$$

<u>Note</u>

$$\tilde{y}^T \tilde{A} = -y^T A$$

(b)

$$P_1 = \begin{cases} 2x_1 + 3x_2 + x_3 \le 5 \\ 3x_1 + 4x_2 + 2x_3 \le 8 \\ -x_1 \le 0 \\ -x_2 \le 0 \\ -x_3 \le 0 \end{cases} = \underbrace{\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 2 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \le \underbrace{\begin{bmatrix} 5 \\ 8 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{b}$$

$$P_2 = \begin{cases} -5x_1 - 4x_2 - 3x_3 \le -14 \\ 4x_1 + x_2 + 2x_3 \le 11 \\ -x_1 \le 0 \\ -x_2 \le 0 \\ -x_3 \le 0 \end{cases} = \underbrace{\begin{bmatrix} -5 & -4 & -3 \\ 4 & 1 & 2 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \le \underbrace{\begin{bmatrix} -14 \\ 11 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\tilde{b}}$$

Then proceed as in (a).