1

Introduction

In this short chapter, we shall explain what is meant by linear programming and sketch a history of this subject.

A DIET PROBLEM

Polly wonders how much money she must spend on food in order to get all the energy (2,000 kcal), protein (55 g), and calcium (800 mg) that she needs every day. (For iron and vitamins, she will depend on pills. Nutritionists would disapprove, but the introductory example ought to be simple.) She chooses six foods that seem to be cheap sources of the nutrients; her data are collected in Table 1.1.

Table 1.1 Nutritive Value per Serving

Food	Serving size	Energy (kcal)	Protein (g)	Calcium (mg)	Price per serving (cents)
Oatmeal	28 g	110	4	2	3
Chicken	100 g	205	32	12	24
Eggs	2 large	160	13	54	13
Whole milk	237 cc	160	8	285	9
Cherry pie	170 g	420	4	22	20
Pork with beans	260 g	260	14	80	19

Then she begins to think about her menu. For example, 10 servings of pork with beans would take care of all her needs for only (?) \$1.90 per day. On the other hand, 10 servings of pork with beans is a lot of pork with beans—she would not be able to stomach more than 2 servings a day. She decides to impose servings-per-day limits on all six foods:

Oatmeal	at most 4 servings per day
Chicken	at most 3 servings per day
Eggs	at most 2 servings per day
Milk	at most 8 servings per day
Cherry pie	at most 2 servings per day
Pork with beans	at most 2 servings per day.

Now, another look at the data shows Polly that 8 servings of milk and 2 servings of cherry pie every day will satisfy the requirements nicely and at a cost of only \$1.12. In fact, she could cut down a little on the pie or the milk or perhaps try a different combination. But so many combinations seem promising that one could go on and on, looking for the best one. Trial and error is not particularly helpful here. To be systematic, we may speculate about some as yet unspecified menu consisting of x_1 servings of oatmeal, x_2 servings of chicken, x_3 servings of eggs, and so on. In order to stay below the upper limits, that menu must satisfy

$$\begin{aligned} 0 &\le x_1 \le 4 \\ 0 &\le x_2 \le 3 \\ 0 &\le x_3 \le 2 \\ 0 &\le x_4 \le 8 \\ 0 &\le x_5 \le 2 \\ 0 &\le x_6 \le 2. \end{aligned} \tag{1.1}$$

And, of course, there are the requirements for energy, protein, and calcium; they lead to the inequalities

$$\begin{aligned} 110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 &\ge 2{,}000 \\ 4x_1 + 32x_2 + 13x_3 + 8x_4 + 4x_5 + 14x_6 &\ge 55 \\ 2x_1 + 12x_2 + 54x_3 + 285x_4 + 22x_5 + 80x_6 &\ge 800. \end{aligned} \tag{1.2}$$

If some numbers $x_1, x_2, \ldots, x_6$ satisfy inequalities (1.1) and (1.2), then they describe a satisfactory menu; such a menu will cost, in cents per day,

$$3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6. \tag{1.3}$$

In designing the most economical menu, Polly wants to find numbers $x_1, x_2, \ldots, x_6$ that satisfy (1.1) and (1.2), and make (1.3) as small as possible. As a mathematician would put it, she wants to

$$
\begin{array}{ll}
\text{minimize} & 3x_1 + 24x_2 + 13x_3 + 9x_4 + 20x_5 + 19x_6 \\
\text{subject to} & 0 \le x_1 \le 4 \\
& 0 \le x_2 \le 3 \\
& 0 \le x_3 \le 2 \\
& 0 \le x_4 \le 8 \\
& 0 \le x_5 \le 2 \\
& 0 \le x_6 \le 2 \\
\multicolumn{2}{l}{110x_1 + 205x_2 + 160x_3 + 160x_4 + 420x_5 + 260x_6 \ge 2000} \\
\multicolumn{2}{l}{\quad 4x_1 + \;\; 32x_2 + \;\; 13x_3 + \quad 8x_4 + \quad 4x_5 + \;\; 14x_6 \ge \quad 55} \\
\multicolumn{2}{l}{\quad 2x_1 + \;\; 12x_2 + \;\; 54x_3 + 285x_4 + \;\; 22x_5 + \;\; 80x_6 \ge \;\; 800.}
\end{array}
\tag{1.4}
$$

Her problem is known as a *diet problem.*

LINEAR PROGRAMMING

Problems of this kind are called "linear programming problems," or "LP problems" for short; linear programming is the branch of applied mathematics concerned with these problems. Here are other examples:

standard form problem

$$
\begin{array}{lrcl}
\text{maximize} & 5x_1 + 4x_2 + 3x_3 & & \\
\text{subject to} & 2x_1 + 3x_2 + x_3 & \le & 5 \\
& 4x_1 + x_2 + 2x_3 & \le & 11 \\
& 3x_1 + 4x_2 + 2x_3 & \le & 8 \\
& x_1, x_2, x_3 & \ge & 0
\end{array}
\tag{1.5}
$$

(with "$x_1, x_2, x_3 \ge 0$" used as shorthand for "$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$") or

$$
\begin{array}{lrcr}
\text{minimize} & 3x_1 - x_2 & & \\
\text{subject to} & -x_1 + 6x_2 - x_3 + x_4 & \ge & -3 \\
& 7x_2 \qquad\quad + 2x_4 & = & 5 \\
& x_1 + x_2 + x_3 \qquad\quad & = & 1 \\
& x_3 + x_4 & \le & 2 \\
& x_2, x_3 & \ge & 0.
\end{array}
\tag{1.6}
$$

In general, if $c_1, c_2, \ldots, c_n$ are real numbers, then the function f of real variables $x_1, x_2, \ldots, x_n$ defined by

$$f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n = \sum_{j=1}^{n} c_jx_j$$