MA 421 Fall 2024 (Aaron N. K. Yip)

Homework 9, due on Thursday, Oct 31st, in class

1. [V] p.168, Exercises: 10.2, 10.5.

   [C] p.248, Problems: 16.10 – just do the "if" part. (Note that in [C], y is a *row* vector.)

2. Consider [V] p.208, Figure 12.8. The four data points are $(0, 0), (2, 1), (4, 2)$ and $(1, 3)$. It is clear that the point $(1, 3)$ is an outlier and the remaining points lie perfectly on the straight line $y = \frac{1}{2}x$. It is found (by many students in Homework 8) that the best fit L^1-regression line is given by $y = \frac{1}{2}x$, which completely ignores the outlier. Is this true more generally? More precisely, consider the following data points:

$$(0, 0), (2, 1), (4, 2), (1, p).$$

   Prove or disprove: the line $y = \frac{1}{2}x$ is the best fit L^1-regression curve regardless of the value of p.

3. Find the biggest circle that can fit inside the following set:

$$x_2 \le \frac{x_1}{2} + 2, \;\; x_1 + x_2 \le 5, \;\; x_1 \ge 0, \;\; x_2 \ge 0.$$

   Draw the above set and also the circle you have found.

   (Note: the above problem can be formulated as an LP – see [MG] p.23.)

4. (a) Consider the following two polyhedrons:

$$\begin{aligned} P_1 &= \{x_1 \ge 0;\;\; x_2 \ge \frac{x_1}{2} + 5;\;\; x_1 + x_2 \ge 10\}; \\ P_2 &= \{x_2 \ge 0;\;\; x_2 \le x_1;\;\; x_2 \le -3x_1 + 30\}. \end{aligned}$$

   Use Farkas' Lemma to find two disjoint half spaces H_1 and H_2 such that $P_1 \subseteq H_1$ and $P_2 \subseteq H_2$. After you have found your answer, plot P_1, P_2, H_1 and H_2 in a single x_1x_2-plane.

   (b) Consider the following two polyhedrons:

$$P_1 = \left\{ \begin{array}{rl} 2x_1 + 3x_2 + x_3 & \le 5; \\ 3x_1 + 4x_2 + 2x_3 & \le 8; \\ x_1, \;\; x_2, \;\; x_3 & \ge 0 \end{array} \right. \quad \text{and} \quad P_2 = \left\{ \begin{array}{rl} 5x_1 + 4x_2 + 3x_3 & \ge 14; \\ 4x_1 + x_2 + 2x_3 & \le 11; \\ x_1, \;\; x_2, \;\; x_3 & \ge 0 \end{array} \right.$$

   Use Farkas' Lemma to find two disjoint half spaces H_1 and H_2 such that $P_1 \subseteq H_1$ and $P_2 \subseteq H_2$.

5. This is Problem #4 from the Midterm. A solution, using completely the algebraic approach, is posted in the course webpage. Redo this problem by using purely a graphical approach, by plotting the feasible set and relevant functions in an x_1x_2-plane.

Consider the following linear programming problem:

$$\begin{array}{lrl} \text{maximize} & x_1 + 2x_2 & \\ \text{subject to} & x_1 + x_2 & \leq 5 \\ & x_1 & \leq 4 \\ & x_2 & \leq 3 \\ & x_1, x_2 & \geq 0 \end{array}$$

(a) Solve the above problem.

(b) Suppose the constraint $x_1 + x_2 \leq 5$ is changed to $x_1 + x_2 \leq p$. Find the range of p such that the dictionary you have found in (a) remains optimal.

(c) Suppose the objective function $x_1 + 2x_2$ is changed to $x_1 + qx_2$. Find the range of q such that the dictionary you have found in (a) remains optimal.