Examples of Partial Differential Equations

(I) (Spatial) Laplace Operator Δ

$$u = u(x, y, z)$$

$$\boxed{\Delta u = \partial_x^2 u + \partial_y^2 u + \partial_z^2 u}$$

$$\boxed{\Delta u = \operatorname{div} \nabla u \quad (= \nabla \cdot \nabla u, \ \nabla^2 u)}$$

$$\nabla u = (\partial_x u, \ \partial_y u, \ \partial_z u)$$

(II) Divergence of a Vector Field

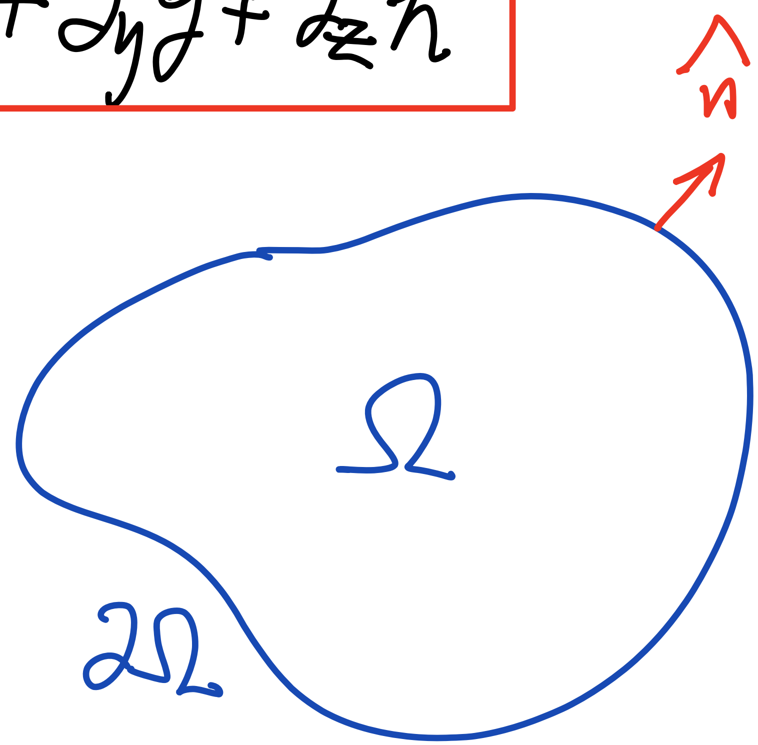
$$F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$F(x,y,z) = (f(x,y,z), g(x,y,z), h(x,y,z))$$

$$\boxed{\operatorname{div} F \; (= \nabla \cdot F) = \partial_x f + \partial_y g + \partial_z h}$$

(III) Divergence Theorem

$$\boxed{\int_{\partial\Omega} F \cdot \hat{n} \, d\sigma = \int_{\Omega} \operatorname{div} F \, dx\,dy\,dz}$$



Examples of PDEs

(1) (1st order) Transport Equation

$$u_t + c\,u_x = 0, \quad c = \text{propagation speed}$$

(2) Heat/Diffusion Equation

$$u_t = D\Delta u = \operatorname{div}(D\nabla u)$$

D (constant) = diffusion coefficient

(3) Wave Equation

$$u_{tt} = c^2 \Delta u, \quad c = \text{propagation speed}$$

Examples of PDEs

(4) Laplace Equation

$$\Delta u = 0$$

(5) Poisson Equation

$$\Delta u = f$$

(e.g. u = electric potential

f = charge density)

Examples of PDEs

(6) Schrödinger Equation

("Newton's 2^{nd} Law in quantum mechanics")

$$\boxed{i\hbar u_t = -\frac{\hbar^2}{2m}\Delta u + V(x)u}$$

(7) Incompressible Navier-Stokes Equation

$$\boxed{\begin{gathered} u_t + u\cdot\nabla u + \nabla p = \sigma \Delta u \\ \operatorname{div} u = 0 \end{gathered}}$$

Examples of PDEs

(6) Schrödinger Equation

("Newton's 2nd Law in quantum mechanics")

$$\boxed{i\hbar u_t = -\frac{\hbar^2}{2m}\Delta u + V(x)u}$$

(7) Stokes Equation (small Reynolds number)

$$\boxed{\begin{aligned} \nabla p &= \sigma \Delta u \\ \operatorname{div} u &= 0 \end{aligned}}$$

Examples of PDEs

(8) Maxwell Equation

J electric current

charge density

$$\begin{cases} \dfrac{\partial E}{\partial t} = c\nabla \times B - 4\pi J; & \operatorname{div} E = 4\pi\rho; \\[2ex] \dfrac{\partial B}{\partial t} = -c\nabla \times E; & \operatorname{div} B = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = \operatorname{div} J \\[2ex] \partial_t^2 E = c^2 \Delta E - 4\pi\left(c^2 \nabla \rho + \partial_t J\right) \\[1ex] \partial_t^2 B = c^2 \Delta B + 4\pi c \nabla \times J \end{cases}$$

Examples of PDEs

(8) Maxwell Equation

$$\begin{cases} \dfrac{\partial E}{\partial t} = c\nabla\times B - 4\pi J; & \operatorname{div} E = 4\pi\rho; \\[2ex] \dfrac{\partial B}{\partial t} = -c\,\nabla\times E; & \operatorname{div} B = 0 \end{cases}$$

if $J \equiv 0$ $\rho \equiv 0$

$$\begin{cases} \dfrac{\partial \rho}{\partial t} = 0 \\[2ex] \left.\begin{aligned} \partial_t^2 E &= c^2 \Delta E \\ \partial_t^2 B &= c^2 \Delta B \end{aligned}\right\} \text{ Wave equations for } E, B \end{cases}$$