Solution of Linear Differential Equations

1st order (homogeneous)

$$\dot{x}(t) = a\,x(t), \quad x(0) = x_0$$

$$\boxed{x(t) = x_0 e^{at}}$$

$(a = \text{constant})$

1st order (inhomogeneous)

$$\dot{x}(t) = a\,x(t) + b(t), \quad x(0) = x_0$$

$$\boxed{x(t) = x_0 e^{at} + \int_0^t e^{a(t-s)} b(s)\, ds}$$

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particular solution

homog. solution

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$$\boxed{x(t) = x_0\, e^{\int_0^t a(s)\,ds}} \qquad (a = a(t))$$

1st order (inhomogeneous)

$$\dot{x}(t) = a\,x(t) + b(t), \quad x(0) = x_0$$

$$\boxed{x(t) = x_0\, e^{\int_0^t a(r)\,dr} + \int_0^t e^{\int_s^t a(r)\,dr}\, a(s)\,ds}$$

Solution of Linear Differential Equations

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$$\boxed{x(t) = x_0\, e^{\int_0^t a(s)\,ds}}$$

$(a = a(t))$

1^{st} order (inhomogeneous)

$$\dot{x}(t) = a\,x(t) + b(t), \quad x(0) = x_0$$

homog. solution

particular solution

$$\boxed{x(t) = x_0\, e^{\int_0^t a(r)\,dr} + \int_0^t e^{\int_s^t a(r)\,dr}\, a(s)\,ds}$$

2^{nd} Order Constant Coefficients (Homog.)

[F. p. 13, Thm 1.1]

$$\begin{cases} \ddot{x} + a\dot{x} + bx = 0 \\ x(0) = p, \quad \dot{x}(0) = q \end{cases}$$

(a, b – constant)

Characteristic polynomial

$$r^2 + ar + b = 0$$

$$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2} = r_1, r_2$$

2nd Order Constant Coefficients (Homog.)

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$$\begin{cases} \ddot{x} + a\dot{x} + bx = 0 \\ x(0) = p, \quad \dot{x}(0) = q \end{cases}$$

(a, b – constant)

Characteristic polynomial

$r_1 \neq r_2$, real numbers

$$\boxed{x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}}$$

(found using p, q)

2^{nd} Order Constant Coefficients (Homog.)

[F. p. 13, Thm 1.1]

$$\begin{cases} \ddot{x} + a\dot{x} + bx = 0 \\ x(0) = p, \quad \dot{x}(0) = q \end{cases}$$

(a, b – constant)

Characteristic polynomial

$$r_1 = r_2 = r \quad \text{(repeat roots)}$$

$$\boxed{x(t) = c_1 e^{rt} + c_2\, t e^{rt}}$$

(found using p, q)

2nd Order Constant Coefficients (Homog.)

[F. p. 13, Thm 1.1]

$$\begin{cases} \ddot{x} + a\dot{x} + bx = 0 \\ x(0) = p, \quad \dot{x}(0) = q \end{cases}$$

(a, b – constant)

Characteristic polynomial

$r_1, r_2 = \alpha \pm \beta i$ (Complex roots

$$\boxed{x(t) = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t}$$

found using p, q.

2^{nd} Order Inhomogeneous

$$\ddot{x} + a\dot{x} + bx = f(t)$$ (a, b can depend on t.)

Let $\varphi_1(t)$ and $\varphi_2(t)$ be 2 homogeneous solutions.

(i.e. the general solution of $\ddot{x} + a\dot{x} + bx = 0$

is given by $x(t) = c_1\varphi_1(t) + c_2\varphi_2(t)$)

Then, a particular solution $x_p(t)$ is given by

$$\boxed{x_p(t) = -\varphi_1(t)\int_0^t \frac{\varphi_2(s) f(s)}{W(s)}\,ds + \varphi_2(t)\int_0^t \frac{\varphi_1(s) f(s)}{W(s)}\,ds}$$

2nd Order Inhomogeneous

$$\ddot{x} + a\dot{x} + b x = f(t)$$ (a, b can depend on t.)

Let $\varphi_1(t)$ and $\varphi_2(t)$ be 2 homogeneous solutions.
(ie. the general solution of $\ddot{x} + a\dot{x} + bx = 0$
is given by $x(t) = c_1\varphi_1(t) + c_2\varphi_2(t)$)

Then, a particular solution $x_p(t)$ is given by

where
$$W(t) = \varphi_1(t)\dot{\varphi}_2(t) - \varphi_2(t)\dot{\varphi}_1(t)$$ is the Wronskian.

2nd Order Inhomogeneous

$$\ddot{x} + a\dot{x} + bx = f(t)$$ (a, b can depend on t.)

Let $\varphi_1(t)$ and $\varphi_2(t)$ be 2 homogeneous solutions.
(i.e. the general solution of $\ddot{x} + a\dot{x} + bx = 0$
is given by $x(t) = c_1\varphi_1(t) + c_2\varphi_2(t)$)

Then, general (inhomogeneous) solution is
given by:

$$\boxed{x(t) = c_1\varphi_1(t) + c_2\varphi_2(t) + x_p(t)}$$

2nd Order Inhomogeneous (Specific Ex.)

[F. p.109 (4.20, 4.21)]

$$\ddot{x} + \alpha^2 x = b(t) \qquad (\alpha > 0)$$

$$\boxed{x(t) = x_h(t) + x_p(t)}$$

$$\boxed{x_h(t) = c_1 \cos \alpha t + c_2 \sin \alpha t}$$

$$\boxed{x_p(t) = \frac{1}{\alpha} \int_0^t (\sin \alpha (t-s))\, b(s)\, ds}$$

2nd Order Inhomogeneous (Specific Ex.)

[F. p.114, #7]

$$\ddot{x} + \alpha^2 x = \beta \sin(kt) \quad (\alpha > 0)$$

$$\boxed{x(t) = x_h(t) + x_p(t)}$$

$$\boxed{x_h(t) = c_1 \cos \alpha t + c_2 \sin \alpha t}$$

$$\boxed{x_p(t) = \frac{\beta \sin(kt)}{\alpha^2 - k^2}, \quad k \neq \alpha}$$

2nd Order Inhomogeneous (Specific Ex.)

[F. p.114, #7]

$$\ddot{x} + \alpha^2 x = \beta \sin(kt) \quad (\alpha > 0)$$

$$\boxed{x(t) = x_h(t) + x_p(t)}$$

$k = \alpha$

$$\boxed{x_h(t) = c_1 \cos \alpha t + c_2 \sin \alpha t}$$

$$\boxed{x_p(t) = -\frac{\beta}{2\alpha} t \cos(\alpha t)}$$

Euler Equation [F. p.117, (4.32)]

$$\boxed{t^2 \ddot{x}(t) + a t \dot{x}(t) + b x(t) = 0}$$

Let $x(t) = t^\lambda$. Then λ satisfies:

$$\boxed{\lambda(\lambda-1) + a\lambda + b = 0, \text{ i.e. } \lambda^2 + (a-1)\lambda + b = 0}$$

$$\lambda = \frac{-(a-1) \pm \sqrt{(a-1)^2 - 4b}}{2} = \lambda_1, \lambda_2$$

Euler Equation [F. p.117, (4.32)]

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Let $x(t) = t^\lambda$. Then λ satisfies:

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If $\lambda_1 \neq \lambda_2$, real,

$$\boxed{x(t) = c_1 t^{\lambda_1} + c_2 t^{\lambda_2}}$$

Euler Equation [F. p.117, (4.32)]

$$\boxed{t^2\ddot{x}(t) + at\dot{x}(t) + bx(t) = 0}$$

Let $x(t) = t^\lambda$. Then λ satisfies:

$$\boxed{\lambda(\lambda-1) + a\lambda + b = 0, \text{ i.e. } \lambda^2 + (a-1)\lambda + b = 0}$$

If $\lambda_1 = \lambda_2 = \lambda$ (repeated roots)

$$\boxed{x(t) = c_1 t^\lambda + c_2 t^\lambda \log t}$$

Euler Equation [F. p.117, (4.32)]

$$\boxed{t^2 \ddot{x}(t) + a t \dot{x}(t) + b x(t) = 0}$$

Let $x(t) = t^\lambda$. Then λ satisfies:

$$\boxed{\lambda(\lambda-1) + a\lambda + b = 0, \text{ i.e. } \lambda^2 + (a-1)\lambda + b = 0}$$

If $\lambda_1, \lambda_2 = \alpha \pm \beta i$, (Complex roots)

$$\boxed{x(t) = C_1 t^\alpha \cos(\beta \log t) + C_2 t^\alpha \sin(\beta \log t)}$$