Separation of Variables — Vector Case

$X = X(t) \in \mathbb{R}^n$ solves :

$$\boxed{\alpha \ddot{X}(t) + \beta \dot{X}(t) = A X(t)} \qquad \left(\dot{} = \frac{d}{dt} \right)$$

A = constant matrix, eg. $n=2$, $A = \begin{pmatrix} -5 & 2 \\ -6 & 2 \end{pmatrix}$

Examples :

(1) $\alpha = 0, \ \beta = 1$: $\dot{X}(t) = A X(t)$ ("Heat eqn")

(2) $\alpha = 1, \ \beta = 0$: $\ddot{X}(t) = A X(t)$ ("Wave eqn")

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Set $$\boxed{\vec{X}(t) = T(t)\,\vec{V}}$$

time dependent scalar function

time independent Vector

Separation of Variables — Vector Case

$$\boxed{\alpha \ddot{X}(t) + \beta \dot{X}(t) = A X(t)}$$

$$\boxed{\vec{X}(t) = T(t) \vec{V}}$$

$$\alpha \ddot{T}(t) V + \beta \dot{T}(t) V = A \left(T(t) \vec{V} \right) = T(t) A V$$

$$\left(\alpha \ddot{T}(t) + \beta \dot{T}(t) \right) V = T(t) A V$$

$$\left(\frac{\alpha \ddot{T}(t) + \beta \dot{T}(t)}{T(t)} \right) V = A V$$

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$$\left(\alpha \ddot{T}(t) + \beta \dot{T}(t)\right) V = T(t) A V$$

$$\left(\frac{\alpha \ddot{T}(t) + \beta \dot{T}(t)}{T(t)}\right) V = A V$$

must be a constant number

Separation of Variables — Vector Case

$$\left(\frac{\alpha \ddot{T}(t) + \beta \dot{T}(t)}{T(t)}\right) V = A V$$

λ = a constant number

(1) $$\boxed{\lambda V = A V}$$

λ – eigenvalue

V = eigenvector of A

(2) $$\frac{\alpha \ddot{T} + \beta \dot{T}}{T} = \lambda \Rightarrow \boxed{\alpha \ddot{T} + \beta \dot{T} = \lambda T}$$

Separation of Variables — Vector Case

(1) $$\boxed{\lambda V = AV}$$

λ – eigenvalue $\qquad V$ = eigenvector of A

eg: $A = \begin{pmatrix} -5 & 2 \\ -6 & 2 \end{pmatrix}$

(i) $$\begin{pmatrix} -5 & 2 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = -1\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad \lambda_1, V_1$$

(ii) $$\begin{pmatrix} -5 & 2 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = -2\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \qquad \lambda_2, V_2$$

Separation of Variables — Vector Case

(2) $$\frac{\alpha \ddot{T} + \beta \dot{T}}{T} = \lambda \Longrightarrow \boxed{\alpha \ddot{T} + \beta \dot{T} = \lambda T}$$

[F, p. 13, Thm 1.1] Characteristic polynomial:

$$\alpha r^2 + \beta r - \lambda = 0 \Longrightarrow r = r_1, r_2$$

$$T(t) = \begin{cases} c_1 e^{r_1 t} + c_2 e^{r_2 t} & r_1 \neq r_2, \text{ real} \\ c_1 e^{r_1 t} + c_2 t e^{r_1 t} & r_1 = r_2 \\ c_1 e^{at} \cos bt + c_2 e^{at} \sin \beta t & r_1, r_2 = a \pm bi \end{cases}$$

Separation of Variables — Vector Case

(2) $$\frac{\alpha \ddot{T} + \beta \dot{T}}{T} = \lambda \Rightarrow \boxed{\alpha \ddot{T} + \beta \dot{T} = \lambda T}$$

[F, p. 13, Thm 1.1] Characteristic polynomial:

$$\alpha r^2 + \beta r - \lambda = 0 \Rightarrow r = r_1, r_2$$

$$\boxed{T(t) = c_1 T_1(t) + c_2 T_2(t)}$$

Separation of Variables — Vector Case

(2) $$\frac{\alpha \ddot{T} + \beta \dot{T}}{T} = \lambda \Rightarrow \boxed{\alpha \ddot{T} + \beta \dot{T} = \lambda T}$$

(i) $\lambda = \lambda_1 \Rightarrow$

$$T(t) = c_1^{(1)} T_1^{(1)}(t) + c_2^{(1)} T_2^{(1)}(t)$$

(ii) $\lambda = \lambda_2 \Rightarrow$

$$T^{(2)}(t) = c_1^{(2)} T_1^{(2)}(t) + c_2^{(2)} T_2^{(2)}(t)$$

Separation of Variables — Vector Case

$$\boxed{\alpha \ddot{X}(t) + \beta \dot{X}(t) = A X(t)}$$

$$\boxed{\vec{X}(t) = T(t) \vec{V}}$$

General Solution : (Superposition)

$\lambda = \lambda_1$

$\lambda = \lambda_2$

$$X(t) = \left(c_1^{(1)} T_1^{(1)}(t) + c_2^{(1)} T_2^{(1)}(t)\right) V_1 + \left(c_1^{(2)} T_1^{(2)}(t) + c_2^{(2)} T_2^{(2)}(t)\right) V_2$$

Example

$$\dot{X}(t) = A\,X(t), \qquad X(0) = \begin{pmatrix}1\\1\end{pmatrix}$$

1st order, $\alpha = 0$, $\beta = 1$

$$\dot{T} = \lambda T \implies r = \lambda$$

$$\implies T(t) = Ce^{\lambda t}$$

$$\boxed{\begin{aligned} X(t) &= c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2 \\ &= c_1 e^{-t}\begin{pmatrix}1\\2\end{pmatrix} + c_2 e^{-2t}\begin{pmatrix}2\\3\end{pmatrix} \end{aligned}}$$

Example

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initial condition

Example

$$\ddot{X}(t) = A X(t), \quad X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \dot{X}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

2nd order, $\alpha = 1$, $\beta = 0$

$$\ddot{T} = \lambda T$$

$$\Longrightarrow \quad r^2 = \lambda = \begin{cases} -1 \\ -2 \end{cases}, \quad r = \begin{cases} \pm i \\ \pm \sqrt{2}\, i \end{cases}$$

$$T(t) = \begin{cases} c_1 \cos t + c_2 \sin t & \leftarrow \lambda_1 = -1 \\ c_3 \cos\sqrt{2}t + c_4 \sin\sqrt{2}t & \leftarrow \lambda_2 = -2 \end{cases}$$

Example

$$\ddot{X}(t) = A X(t), \quad X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \dot{X}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

2nd order, $\alpha = 1$, $\beta = 0$

$$\boxed{\begin{aligned} X(t) &= (c_1 \cos t + c_2 \sin t)\begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &\quad + (c_3 \cos\sqrt{2}t + c_4 \sin\sqrt{2}t)\begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{aligned}}$$

Example

$$\ddot{X}(t) = A X(t), \quad X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \dot{X}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

2nd order, $\alpha = 1$, $\beta = 0$

initial conditions

$$\boxed{X(t) = (c_1 \cos t + c_2 \sin t)\begin{pmatrix} 1 \\ 2 \end{pmatrix} + (c_3 \cos\sqrt{2}t + c_4 \sin\sqrt{2}t)\begin{pmatrix} 2 \\ 3 \end{pmatrix}}$$