Proof of Bessel Inequality (using complex no.)

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

Bessel Inequality (F, p.30)

$$\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$

(Parseval Equality (F. p.77)

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$)

Pf

$$f(x) = \underbrace{f(x) - \sum_{n=-N}^{N} c_n e^{inx}}_{e_N(x) = \text{error}} + \sum_{n=-N}^{N} c_n e^{inx}$$

$$|f(x)|^2 = \left(\sum_{n=-N}^{N} c_n e^{inx} + e_N(x)\right) \cdot \overline{\left(\sum_{m=-N}^{N} c_m e^{imx} + e_N(x)\right)}$$

$$= \left(\sum_{n=-N}^{N} c_n e^{inx} + e_N(x)\right)\left(\sum_{m=-N}^{N} \overline{c_m} e^{-imx} + \overline{e_N(x)}\right)$$

$$\left(\overline{e^{imx}} = e^{-imx}\right)$$

change the dummy index

$$|f(x)|^2 = \left(\sum_{n=-N}^{N} c_n e^{inx} + e_N(x)\right)\left(\sum_{m=-N}^{N} \overline{c_m}\, e^{-imx} + \overline{e_N(x)}\right)$$

$$= \sum_{n=-N}^{N} \sum_{m=-N}^{N} c_n \overline{c_m}\, e^{inx} e^{-imx}$$

$$+ \; e_N(x) \sum_{m=-N}^{N} \overline{c_m}\, e^{-imx}$$

$$+ \; \overline{e_N(x)} \sum_{m=-N}^{N} c_n e^{inx}$$

$$+ \; |e_N(x)|^2$$

$$\int_{-\pi}^{\pi} |f(x)|^2 = \left(\sum_{n=-N}^{N} c_n e^{inx} + e_N(x)\right)\left(\sum_{m=-N}^{N} \overline{c_m} e^{-imx} + \overline{e_N(x)}\right)$$

$$= \int_{-\pi}^{\pi} \sum_{n=-N}^{N} \sum_{m=-N}^{N} c_n \overline{c_m} e^{inx} e^{-imx} \qquad ①$$

$$+ \int_{-\pi}^{\pi} e_N(x) \sum_{m=-N}^{N} \overline{c_m} e^{-imx} \qquad ②$$

$$+ \int_{-\pi}^{\pi} \overline{e_N(x)} \sum_{n=-N}^{N} c_n e^{inx} \qquad ③$$

$$+ \int_{-\pi}^{\pi} |e_N(x)|^2 \qquad ④$$

$$① = \sum_{n,m} c_n \overline{c_m} \underbrace{\int_{-\pi}^{\pi} e^{inx} e^{-imx} \, dx}_{= \begin{cases} 2\pi & m=n \\ 0 & m \neq n \end{cases}}$$

$$= \sum_{n=-N}^{N} c_n \overline{c_n} \, 2\pi$$

$$= \sum_{n=-N}^{N} |c_n|^2 \, 2\pi$$

$$② = \int_{-\pi}^{\pi} e_N(x)\left(\sum_{=-N}^{N} \overline{c_m}\, e^{-imx}\right) dx$$

$$= \sum_{m=-N}^{N} \overline{C_m} \int_{-\pi}^{\pi} e_N(x)\, e^{-imx}\, dx$$

$$= \sum_{m=-N}^{N} \overline{C_m} \int_{-\pi}^{\pi} \left(f(x) - \sum_{n=-N}^{N} c_n e^{inx}\right) e^{-imx}\, dx$$

$$= \sum_{m=-N}^{N} \overline{C_m} \underbrace{\int_{-\pi}^{\pi} f(x)\, e^{-imx}\, dx}_{2\pi C_m} - \sum_{n,m} \overline{C_m}\, C_n \int_{-\pi}^{\pi} e^{inx}\, e^{-imx}\, dx$$

$$= \sum_{m=-N}^{N} 2\pi \overline{C_m} C_m - \sum_{m=-N}^{N} \overline{C_m} C_m 2\pi$$

$$= 0$$

$$3 = \overline{2} = 0 \quad \text{and} \quad 4 \geq 0$$

Hence

$$\int_{-\pi}^{\pi} |f(x)|^2 dx \geq 2\pi \sum_{n=-N}^{N} |C_n|^2 \quad \text{for each } N.$$

$$\Longrightarrow \int_{-\pi}^{\pi} |f(x)|^2 dx \geq 2\pi \sum_{n=-\infty}^{\infty} |C_n|^2$$

Using a_n & b_n

$$\sum_{n=-N}^{N} c_n e^{inx} = c_0 + \sum_{n=1}^{\infty}\left(c_n e^{inx} + c_{-n} e^{-inx}\right)$$

$$= c_0 + \sum_{n=1}^{\infty} c_n\left(\cos nx + i\sin nx\right) + c_{-n}\left(\cos nx - i\sin nx\right)$$

$$= c_0 + \sum_{n=1}^{\infty} \underbrace{\left(c_n + c_{-n}\right)}_{a_n}\cos nx + \underbrace{i\left(c_n - c_{-n}\right)}_{b_n}\sin nx$$

$$|a_n|^2 = (c_n + c_{-n})(\overline{c_n} + \overline{c_{-n}})$$

$$= |c_n|^2 + c_n\overline{c_{-n}} + c_{-n}\overline{c_n} + |c_{-n}|^2$$

$$|b_n|^2 = i(c_n - c_{-n})(-i(\overline{c_n} - \overline{c_{-n}}))$$

$$= |c_n|^2 - c_n\overline{c_{-n}} - c_{-n}\overline{c_n} + |c_{-n}|^2$$

Hence $|a_n|^2 + |b_n|^2 = 2(|c_n|^2 + |c_{-n}|^2)$

& $\frac{a_0}{2} = c_0 \Rightarrow \frac{|a_0|^2}{4} = |c_0|^2$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2dx \geq \sum_{n=-\infty}^{\infty}|c_n|^2$$

$$= |c_0|^2 + \sum_{n=1}^{\infty}\left(|c_n|^2 + |c_{-n}|^2\right)$$

$$= \frac{|a_0|^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left(|a_n|^2 + |b_n|^2\right)$$