Convergence of Fourier Series

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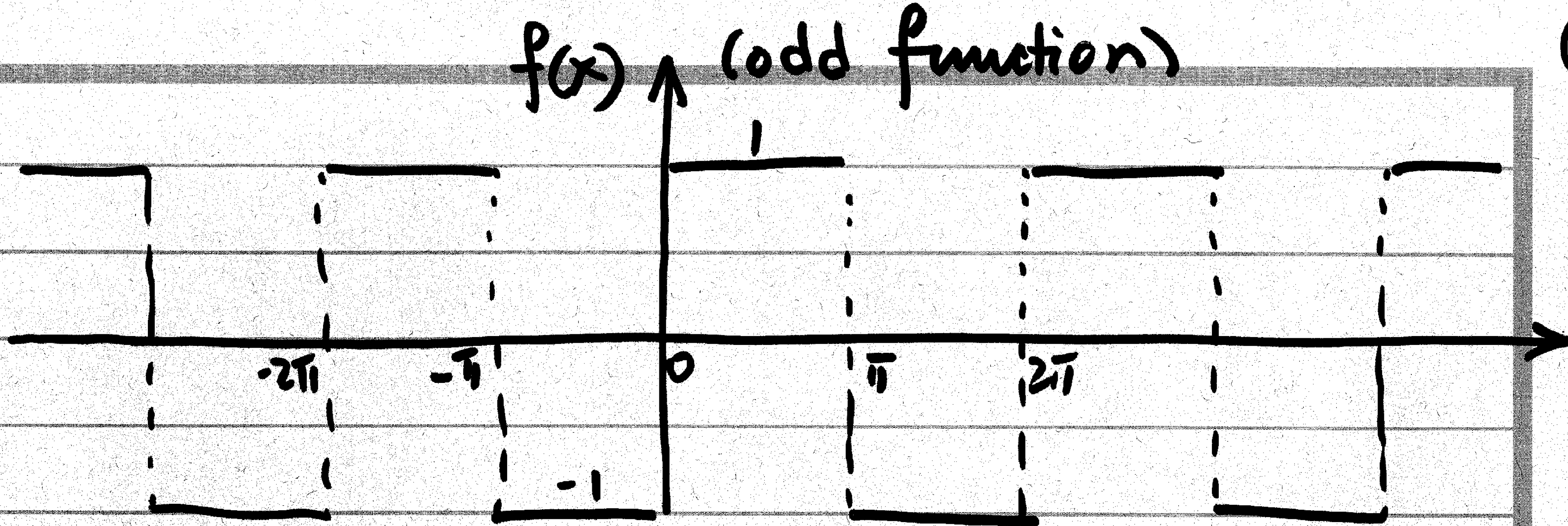
meaning (?)

(2) Nth partial sum :

$$(S_N f)(x) = \sum_{n=-N}^{N} c_n e^{inx}$$

(3) $\lim_{N\to\infty} \left| f(x) - (S_N f)(x) \right| = 0$



$$\boxed{f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)} = \frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right]}$$

Partial sums:

$$\frac{4}{\pi}\sin x \; ; \; \frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3}\right];$$

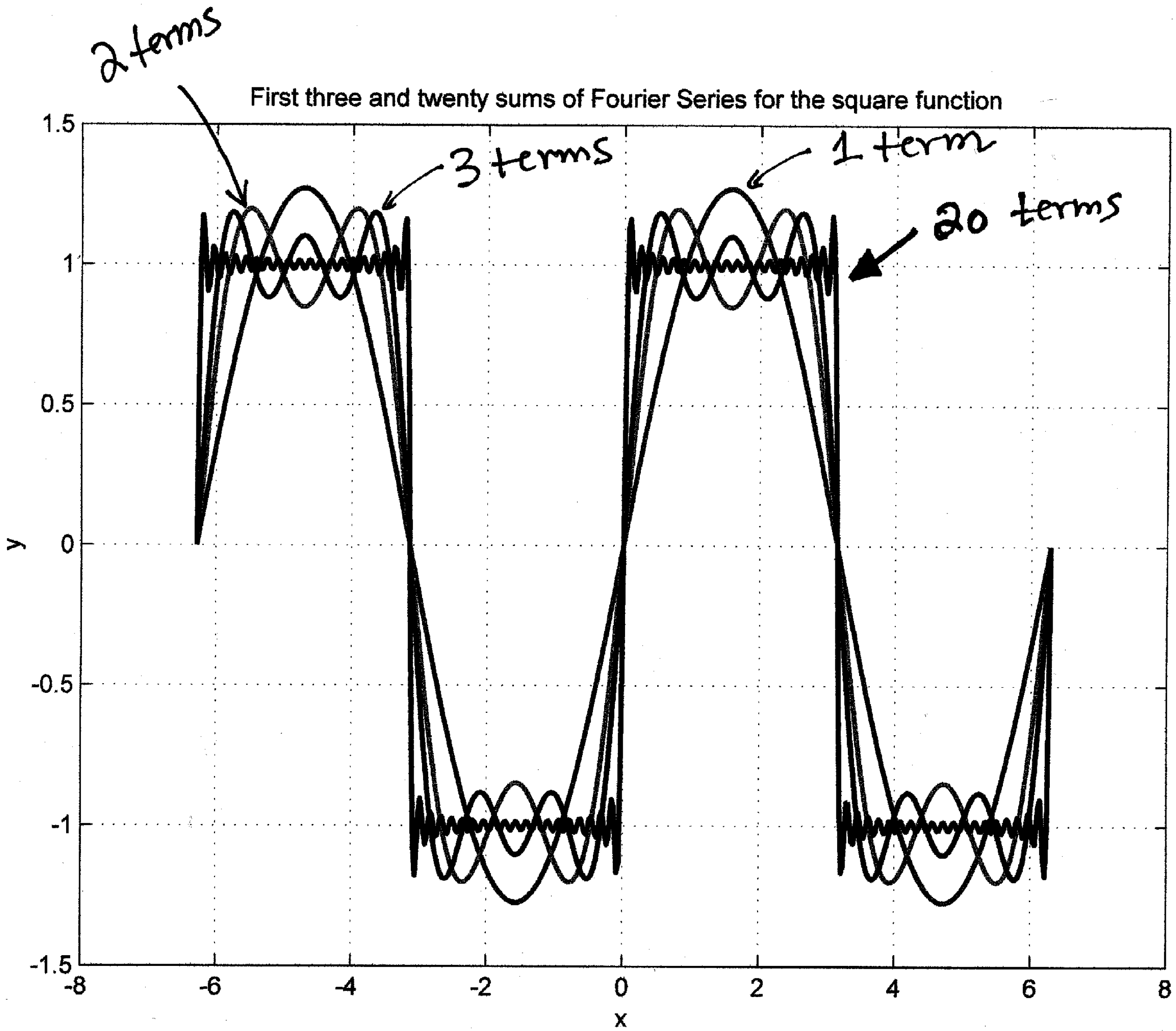
$$\frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5}\right];$$

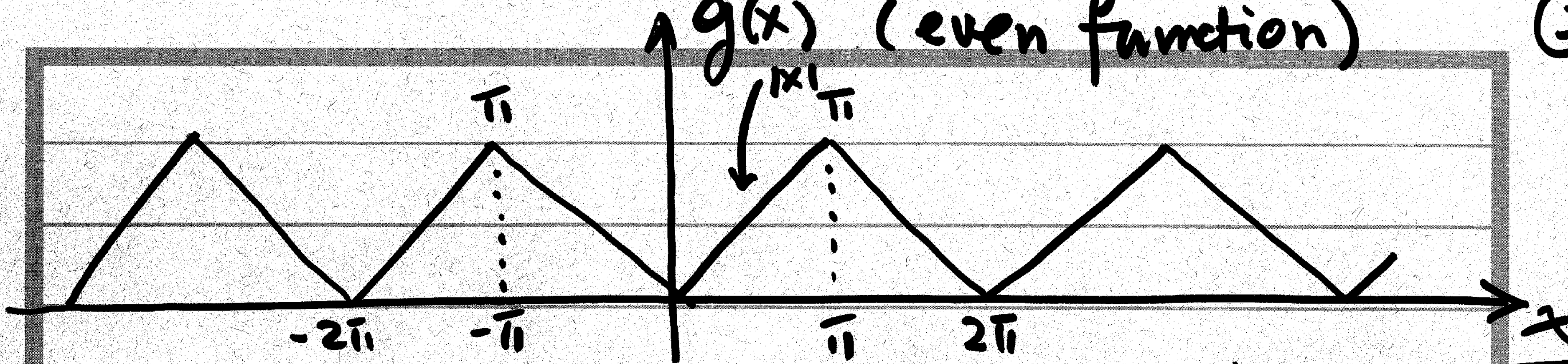
f – piecewise smooth

f – has jumps

f' – does NOT exist

$a_n, b_n = O(1/n)$


2 terms
First three and twenty sums of Fourier Series for the square function
3 terms
1 term
20 terms
1.5
1
0.5
0
-0.5
-1
-1.5
y
-8
-6
-4
-2
0
2
4
6
8
x



$$g(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$$

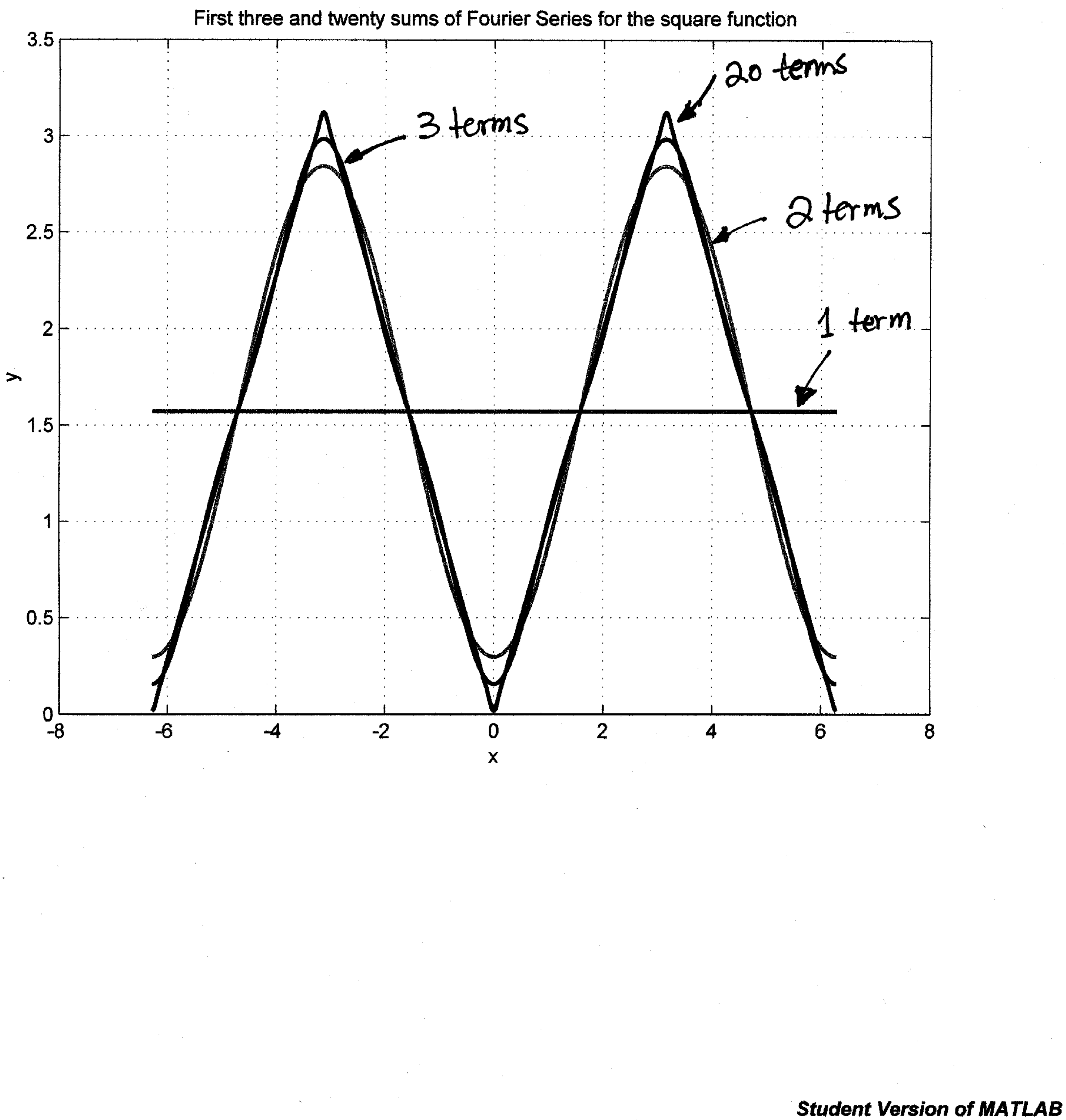
g – no jump
g' – exists
$a_n, b_n \ll O\left(\frac{1}{n}\right)$
$\left(a_n, b_n = o\left(\frac{1}{n}\right)\right)$

Partial Sums:

$$\frac{\pi}{2}\ ; \quad \frac{\pi}{2} - \frac{4}{\pi}\cos x\ ; \quad \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{\pi}\frac{\cos 3x}{3^2}\ ; \ \ldots$$

$$\frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{\pi}\frac{\cos 3x}{3^2} - \frac{4}{\pi}\frac{\cos 5x}{5^2}\ ;$$

(5)


First three and twenty sums of Fourier Series for the square function
3.5
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4
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x
20 terms
3 terms
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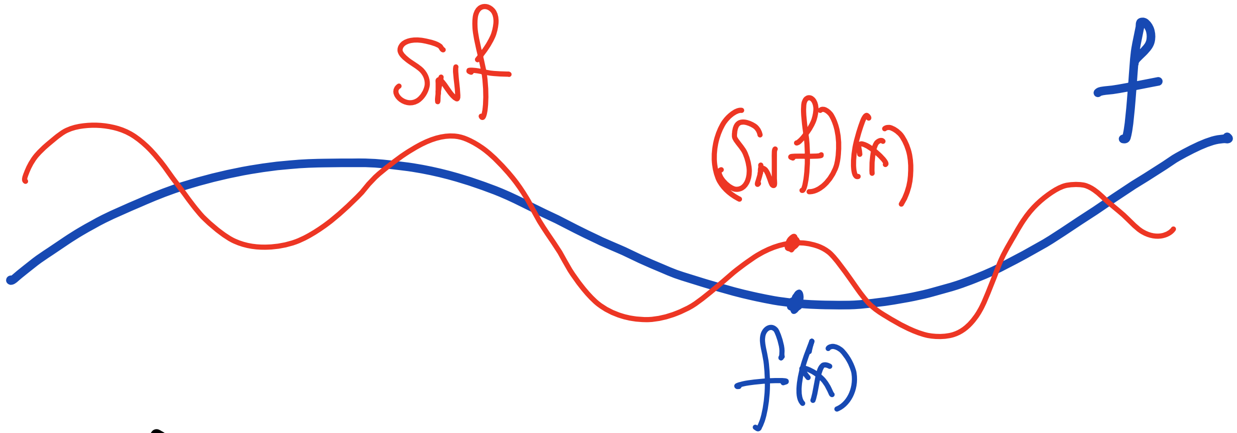
Student Version of MATLAB

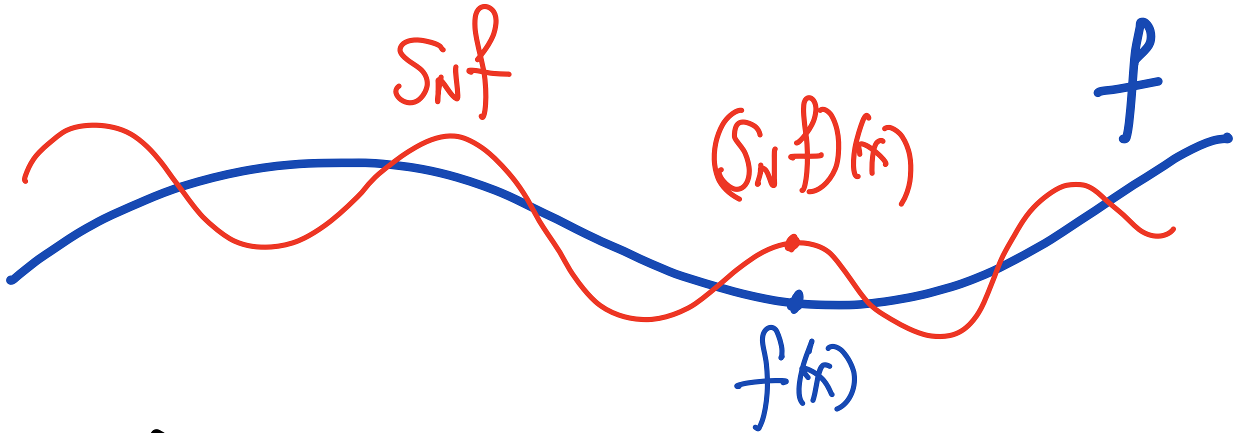
Point-wise Convergence



Def: $S_N f$ converges point-wisely to f at x (fixed) if

$$\boxed{\lim_{N\to\infty} S_N f(x) = f(x)}$$

Point-wise Convergence



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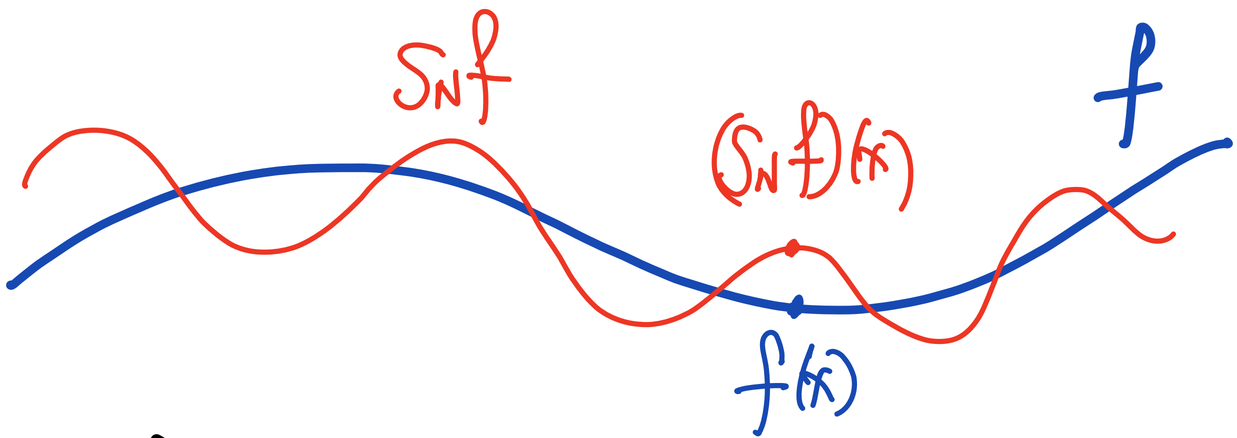
Point-wise Convergence



Def: $S_N f$ converges point-wisely to f at x (fixed) if

$$\boxed{\begin{array}{l}\text{for any } \varepsilon > 0, \text{ there exists } N_\varepsilon(x) \text{ s.t.} \\ \text{for any } N \geq N_\varepsilon(x), \text{ it holds: } |f(x) - S_N f(x)| \leq \varepsilon\end{array}}$$

Dirichlet Kernel

$$(S_N f)(x) = \sum_{n=-N}^{N} c_n e^{inx}$$

$$= \int_{-\pi}^{\pi} f(y) D_N(x-y)\, dy$$

Dirichlet Kernel

$$(S_N f)(x) = \sum_{n=-N}^{N} c_n e^{inx}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) e^{-iny}\, dy$$

$$= \int_{-\pi}^{\pi} f(y) D_N(x-y)\, dy$$

$$D_N(\theta) = \frac{1}{2\pi} \frac{\sin\left(\left(N+\frac{1}{2}\right)\theta\right)}{\sin\left(\theta/2\right)} \quad (-\pi < \theta < \pi)$$

Dirichlet Kernel (Properties)

$$D_N(\theta) = \frac{1}{2\pi} \frac{\sin\left((N+\frac{1}{2})\theta\right)}{\sin(\theta/2)} \qquad -\pi < \theta < \pi$$

(1) $D_N(\cdot)$ is 2π-periodic;

(2) $D_N(\cdot)$ is even function;

(3) $\displaystyle\int_{-\pi}^{\pi} D_N(\theta)\,d\theta = 1$

(4) $\displaystyle D_N(0) = \lim_{\theta \to 0} \frac{\sin\left((N+\frac{1}{2})\theta\right)}{2\pi \sin(\theta/2)} = \frac{(N+\frac{1}{2})}{2\pi}$

Dirichlet Kernel (Properties)

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(5) $D_N(\theta) \sim \frac{1}{2\pi} \sin\left(\left(N+\frac{1}{2}\right)\theta\right)$

when $\theta \sim \pm\pi$

Dirichlet Kernel (Properties)

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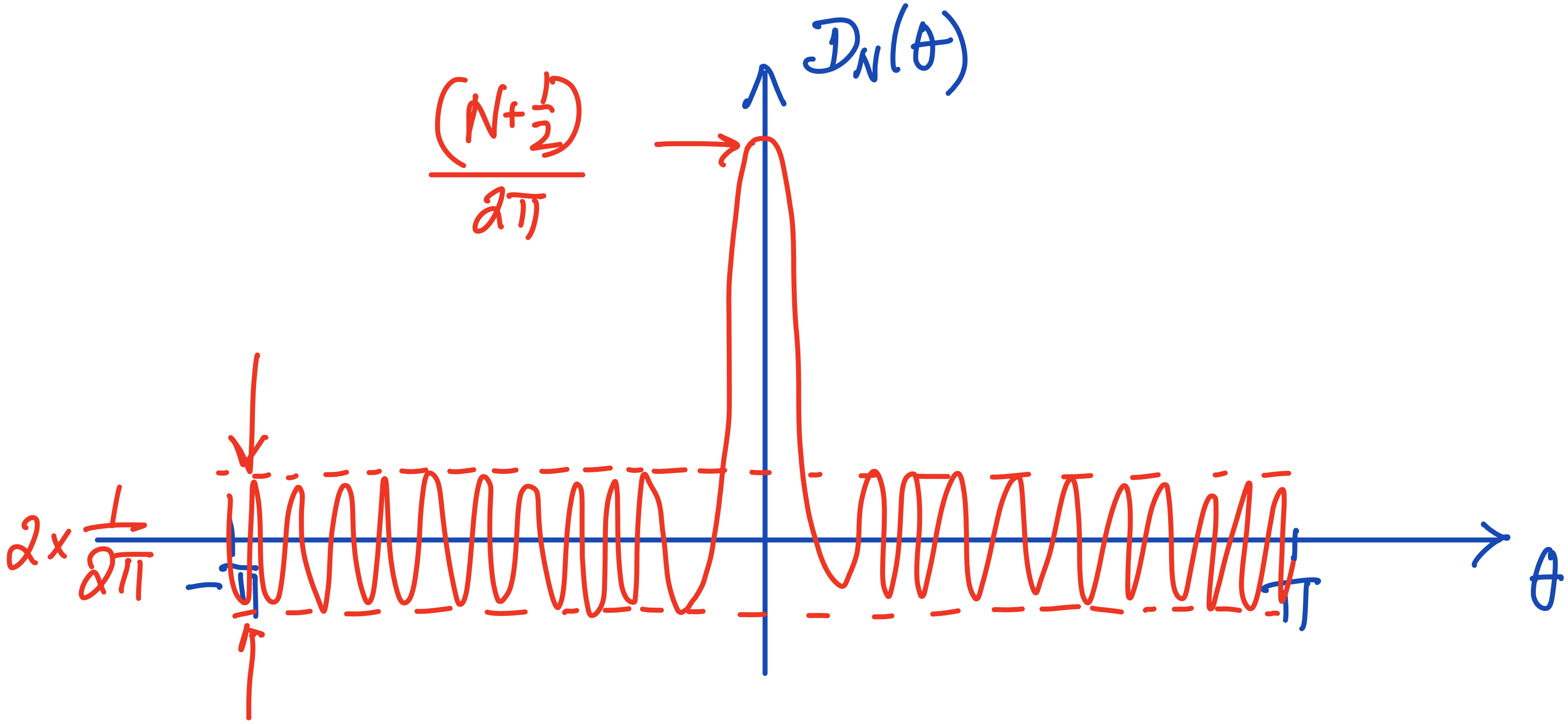

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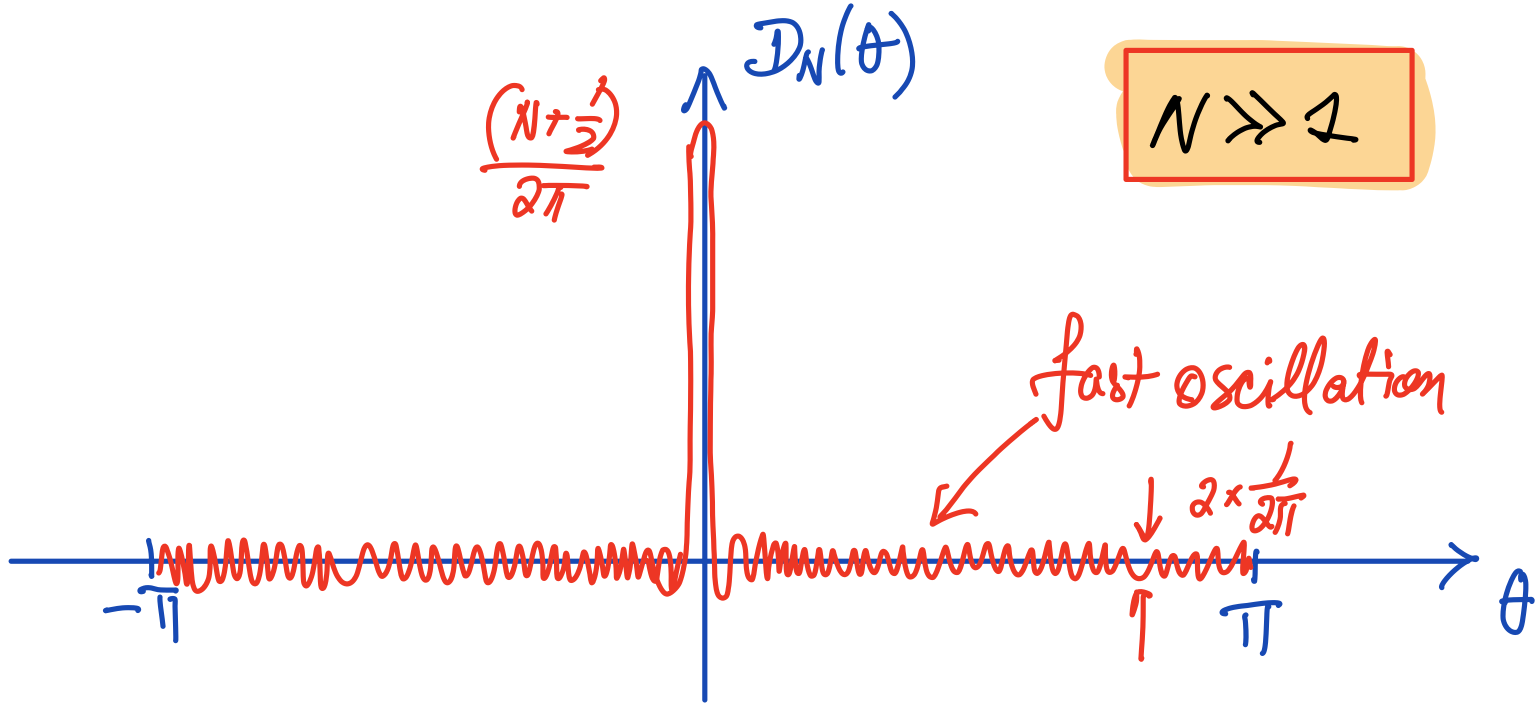

$$\boxed{N \gg 1}$$

Pointwise Convergence & Dirichlet Kernel

$$(S_N f)(x) = \int_{-\pi}^{\pi} f(y) D_N(x-y)\,dy$$

$$= \underbrace{\int f(y) D_N(x-y)\,dy}_{y \text{ near } x} + \underbrace{\int f(y) D_N(x-y)\,dy}_{y \text{ far away from } x}$$

Pointwise Convergence & Dirichlet Kernel

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$$= \int f({\color{red}x}) D_N(x-y)\,dy + \int f(y) D_N(x-y)\,dy$$

Pointwise Convergence & Dirichlet Kernel

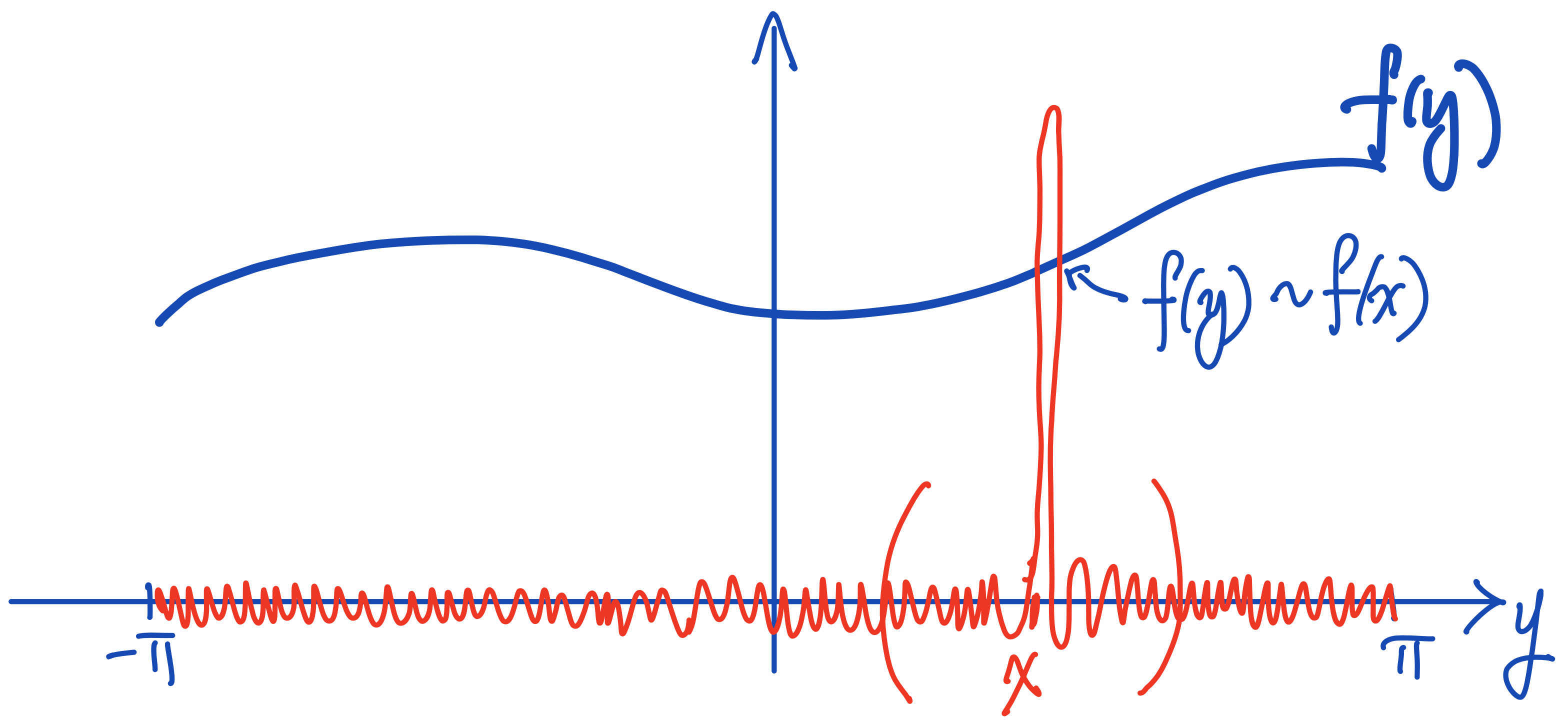
$$(S_N f)(x) = \int_{-\pi}^{\pi} f(y) D_N(x-y)\,dy$$

$$= \int_{y \text{ near } x} f(y) D_N(x-y)\,dy + \int_{y \text{ far away from } x} f(y) D_N(x-y)\,dy$$

$$= \underbrace{\int f(x) D_N(x-y)\,dy}_{\approx f(x)} + \underbrace{\int f(y) D_N(x-y)\,dy}_{\approx 0}$$

Pointwise Convergence & Dirichlet Kernel

$$(S_N f)(x) = \int_{-\pi}^{\pi} f(y) D_N(x-y)\,dy$$

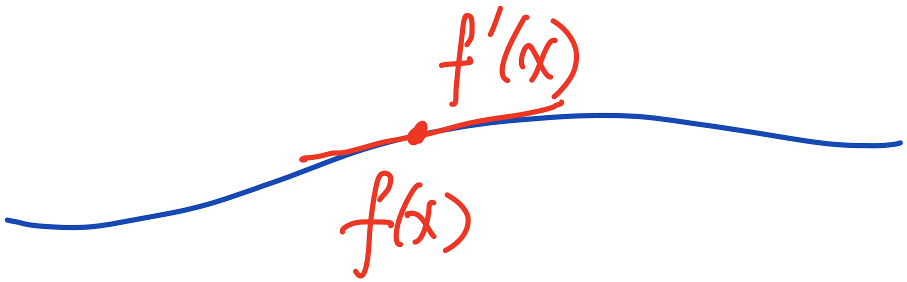


Pointwise Convergence of Fourier Series

(Assume $\int_{-\pi}^{\pi} |f(x)|\,dx < \infty$, i.e. f is integrable)

1. If f is continuous at x and $f'(x)$ exists,

then $$(S_N f)(x) \xrightarrow{N\to\infty} f(x)$$



Theorem 2.1
F, p.35

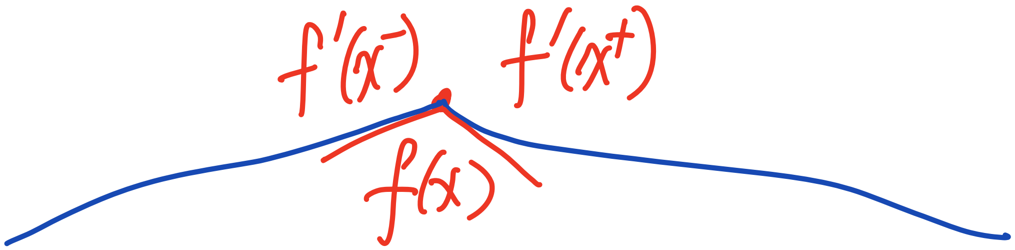
Pointwise Convergence of Fourier Series

(Assume $\int_{-\pi}^{\pi} |f(x)|\,dx < \infty$, i.e. f is integrable)

2

If f is continuous at x and $f'(x^{\pm})$ exist,

then $$(S_N f)(x) \xrightarrow{N\to\infty} f(x)$$



Theorem 2.1
F, p.35

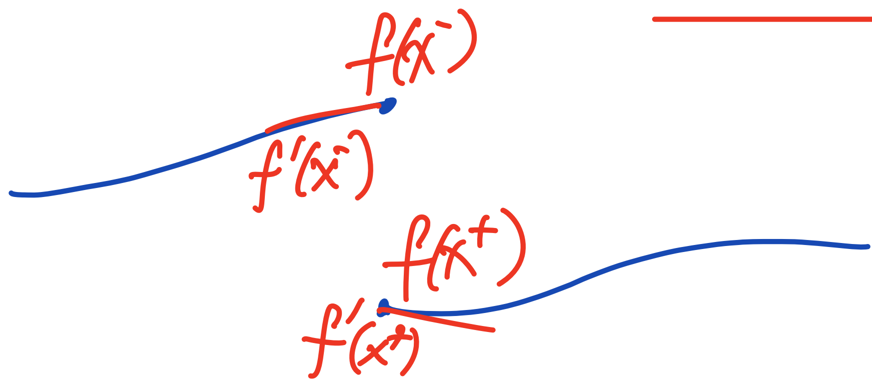
Pointwise Convergence of Fourier Series

(Assume $\int_{-\pi}^{\pi} |f(x)|\,dx < \infty$, i.e. f is integrable)

3

If $f(x^-) \neq f(x^+)$ but $f'(x^-)$ & $f'(x^+)$ exist,

then $(S_N f)(x) \longrightarrow \underline{\frac{1}{2}\left(f(x^-)+f(x^+)\right)}$



Theorem 2.1
F, p.35

Pointwise Convergence of Fourier Series

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③

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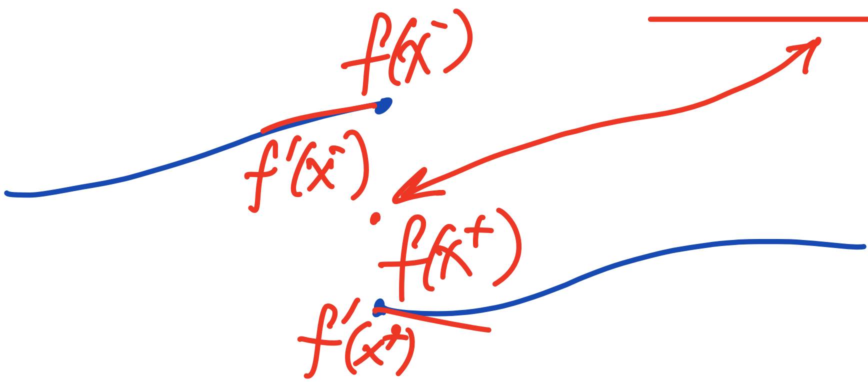
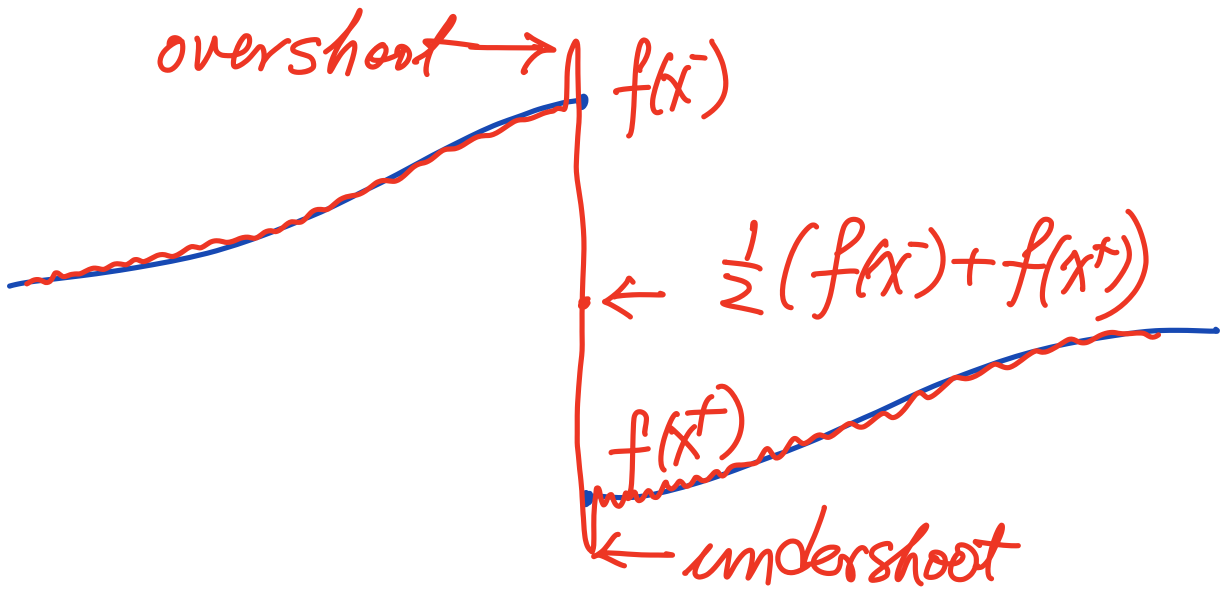


$\boxed{\text{Theorem 2.1} \quad \text{F, p.35}}$

Pointwise Convergence of Fourier Series

(Assume $\int_{-\pi}^{\pi} |f(x)|\,dx < \infty$, i.e. f is integrable)

3 But beware of Gibbs Phenomenon



$\boxed{\text{F, p.62}}$

Uniform Convergence of $S_N f$ to f

Def: $S_N f$ converges uniformly to f if for any ε, there is an N_0 s.t. for any $N \geq N_0$, it holds that

$$|(S_N f)(x) - f(x)| \leq \varepsilon$$

Uniform Convergence of $S_N f$ to f

Def: $S_N f$ converges uniformly to f

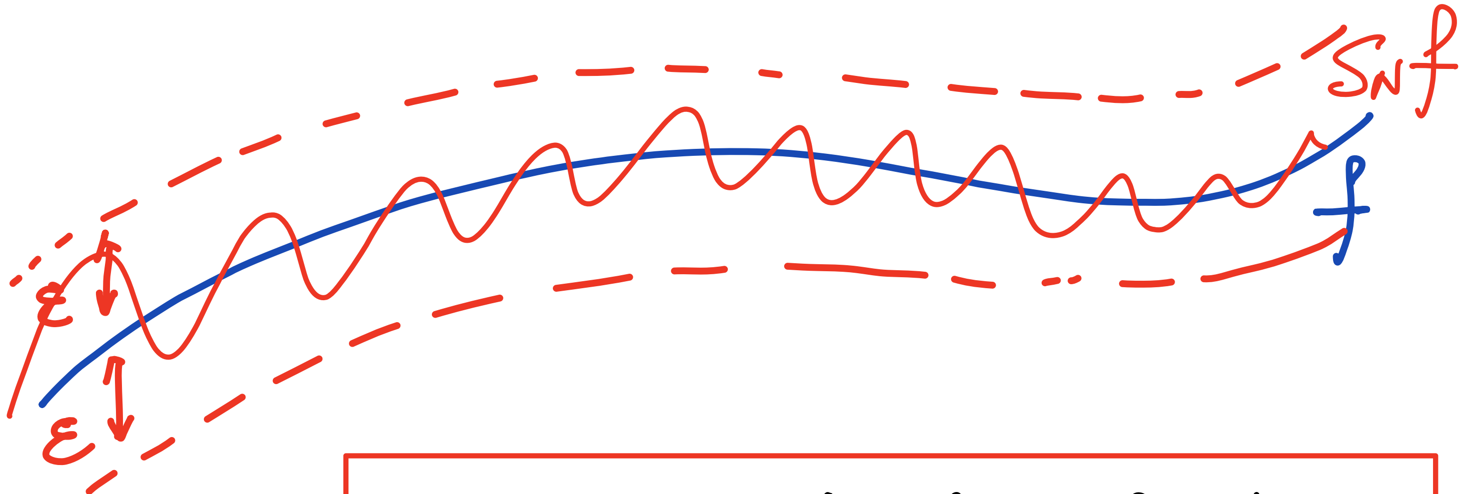
if for any ε, there is an N_0 does not depend on x

s.t. for any $N \geq N_0$, it holds that

$$|(S_N f)(x) - f(x)| \leq \varepsilon \quad \text{for any } x$$

Uniform Convergence of $S_N f$ to f

Def: $S_N f$ converges uniformly to f



$$\boxed{\text{for any } x,\ |S_N f(x) - f(x)| \leq \varepsilon}$$

Uniform Convergence of $S_N f$ to f

Theorem 2.5 (F. p.41)

If f is continuous and piece-wise smooth (i.e. f' exists piecewisely), then $S_N f \longrightarrow f$ uniformly

Uniform Convergence of $S_N f$ to f

Theorem 2.5 (F. p. 41)

If f is continuous and piece-wise smooth (i.e. f' exists piecewisely), then $S_N f \longrightarrow f$ uniformly

(An application of Bessel's Inequality)

$$\int_{-\pi}^{\pi} |f'(x)|^2 dx < \infty$$

Some "Cultural" Education (Korner, Ch. 19)

(1) (Kolmogorov) There is an integrable fct f (i.e. $\int_{-\pi}^{\pi} |f(x)|\,dx < \infty$) such that $S_N f(x)$ does not converge at any x

(2) (Carlson) If f is continuous, then there is some very small set E (measure zero) such that $S_N f(x) \longrightarrow f(x)$ for any $x \notin E$.

Some "Cultural" Education (Korner, Ch. 19)

(3) (Kahne & Katznelson) For any small set E (measure zero), there is a continuous function f s.t.

$$S_N f(x) \longrightarrow f(x)$$

for any $x \notin E$.