# Fourier Series Using Complex Numbers

$f$ – $2\pi$-periodic function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \qquad n \geq 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \qquad n \geq 1$$

# Euler Formula

$$e^{ix} = \cos x + i\sin x$$

$$e^{-ix} = \cos x - i\sin x$$

$$\frac{e^{ix} + e^{-ix}}{2} = \cos x$$

$$\frac{e^{ix} - e^{-ix}}{2i} = \sin x$$

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$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right)$$

$$\frac{e^{inx}+e^{-inx}}{2} \qquad\qquad \frac{e^{inx}-e^{-inx}}{2i}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left( \frac{e^{inx}+e^{-inx}}{2} \right) + b_n \left( \frac{e^{inx}-e^{-inx}}{2i} \right)$$

$$= \underbrace{\frac{a_0}{2}}_{c_0} + \sum_{n=1}^{\infty} \underbrace{\left( \frac{a_n}{2} + \frac{b_n}{2i} \right)}_{c_n,} e^{inx} + \underbrace{\left( \frac{a_n}{2} - \frac{b_n}{2i} \right)}_{c_{-n}} e^{-inx}$$

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$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right)$$

$$= \underbrace{\frac{a_0}{2}}_{c_0} + \sum_{n=1}^{\infty} \underbrace{\left( \frac{a_n}{2} + \frac{b_n}{2i} \right)}_{c_n,} e^{inx} + \underbrace{\left( \frac{a_n}{2} - \frac{b_n}{2i} \right)}_{c_{-n}} e^{-inx}$$

$$= c_0 e^{i\underset{\substack{\uparrow \\ n=0}}{0}x} + \sum_{n=1}^{\infty} \left( c_n e^{inx} + c_{-n} e^{-inx} \right)$$

$$= \boxed{\sum_{n=-\infty}^{\infty} c_n e^{inx}}$$

# Fourier Series Using Complex Numbers

$$c_n = \frac{1}{2}\left(a_n + \frac{b_n}{i}\right) \qquad n>0$$

$$= \frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx + \frac{1}{i}\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\cos nx - i\sin nx\right)dx$$

$$\boxed{= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}\,dx} \qquad n>0$$

# Fourier Series Using Complex Numbers

$$C_{-n} = \frac{1}{2}\left(a_n - \frac{b_n}{i}\right) \qquad n > 0$$

$$= \frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx - \frac{1}{i}\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\cos nx + i\sin nx\right)dx$$

$n < 0$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{+inx}\,dx, \text{ i.e. } \boxed{C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx}$$

# Fourier Series Using Complex Numbers

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

$$\boxed{C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx ,} \quad n = 0, \pm 1, \pm 2, \ldots$$

( Note : for $n=0$,

$$C_0 = \frac{a_0}{2} = \frac{1}{2}\,\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{i0x}\, dx$$ )

# Another Method of Deriving $C_n$

**Note:**

$n \neq m$

$$\int_{-\pi}^{\pi} e^{inx} e^{-imx}\,dx = \int_{-\pi}^{\pi} e^{i(n-m)x}\,dx$$

$$= \left.\frac{e^{i(n-m)x}}{n-m}\right|_{-\pi}^{\pi} = 0$$

$n = m$

$$\int_{-\pi}^{\pi} e^{inx} e^{-\overset{n=m}{inx}}\,dx = \int_{-\pi}^{\pi} 1\,dx = 2\pi$$

# Another Method of Deriving $C_n$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

$\times\, e^{-imx}$

$$\int_{-\pi}^{\pi} f(x)\, e^{-imx} = \sum_{n=-\infty}^{\infty} C_n e^{inx} e^{-imx} \quad dx$$

$$\int_{-\pi}^{\pi} f(x) e^{-imx} dx = \sum_{n=-\infty}^{\infty} C_n \underbrace{\int_{-\pi}^{\pi} e^{i(n-m)x} dx}_{\substack{2\pi \text{ if } n=m \\ 0 \text{ if } n\neq m}}$$

$$= 2\pi C_m$$

# Another Method of Deriving $C_n$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$$

$\times e^{-imx}$

$$\int_{-\pi}^{\pi} f(x)\, e^{-imx} = \sum_{n=-\infty}^{\infty} C_n e^{inx} e^{-imx} \, dx$$

Hence

$$\boxed{C_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} \, dx}$$

( simply change m to n )