Inner Product Space

(1) Inner Product Space

(2) Cauchy-Schwarz Inequality

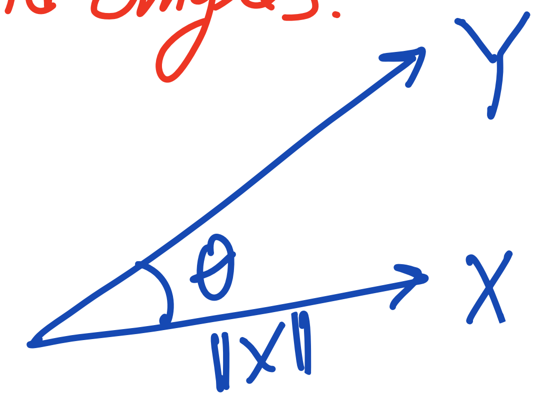
3) Orthogonat Set of Vectors

(4) Projection and Least Square Approx.

(5) Completeness of Basis Vectors

Inner Product Space

To introduce the concept of length and angles.



$\mathbb{R}^3$: $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$X \cdot Y, \ \langle X, Y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$$

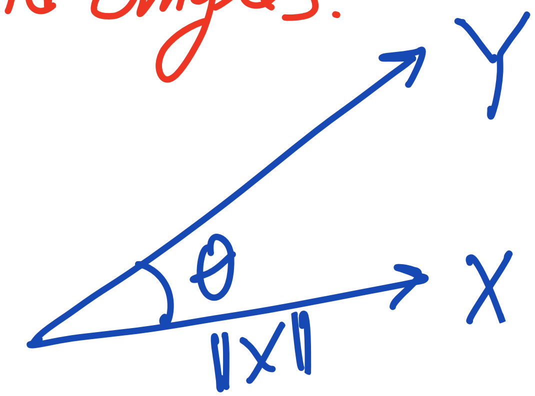
$$\|X\|^2 = \langle X, X \rangle = x_1^2 + x_2^2 + x_3^2$$

$$\|X\| = \sqrt{\langle X, X \rangle} = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

length

Inner Product Space

To introduce the concept of length and angles.



$\mathbb{R}^3$: $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$X \cdot Y, \ \langle X, Y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$$

$$\cos\theta = \frac{\langle X, Y \rangle}{\|X\| \, \|Y\|}, \quad \theta = \cos^{-1}\left(\frac{\langle X, Y \rangle}{\|X\| \, \|Y\|} \right)$$

angle

Inner Product Space

To introduce the concept of length and angles.

Let V be a real vector space.

An inner product for V is:

$$\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{R} \quad \text{satisfying}$$

(1) Symmetry: $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$;

(2) Linearity: $\langle \alpha\vec{u} + \beta\vec{v}, \vec{w} \rangle = \alpha\langle \vec{u}, \vec{w} \rangle + \beta\langle \vec{v}, \vec{w} \rangle$

(3) Positive Definiteness: $\langle \vec{u}, \vec{u} \rangle \geq 0$,

$= 0 \iff \vec{u} = \vec{0}$

Inner Product Space

To introduce the concept of length and angles.

Let $\mathcal{V}$ be a <u>complex</u> vector space.

An <u>inner product</u> for $\mathcal{V}$ is :

$\langle \cdot , \cdot \rangle : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{C}$ satisfying

(1) <u>Symmetry</u>: $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$;

(2) <u>Linearity</u>: $\langle \alpha \vec{u} + \beta \vec{v}, \vec{w} \rangle = \alpha \langle \vec{u}, \vec{w} \rangle + \beta \langle \vec{v}, \vec{w} \rangle$

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Inner Product Space

To introduce the concept of length and angles.

Let V be a complex vector space.

An inner product for V is:

$$\langle \cdot , \cdot \rangle : V \times V \longrightarrow \mathbb{C} \quad \text{satisfying}$$

(1) Symmetry: $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$;

(2) Linearity: $\langle \vec{w}, \alpha\vec{u} + \beta\vec{v} \rangle = \overline{\alpha} \langle \vec{w}, \vec{u} \rangle + \overline{\beta} \langle \vec{w}, \vec{v} \rangle$

(3) Positive Definiteness: $\langle \vec{u}, \vec{u} \rangle \geq 0$,

$= 0 \iff \vec{u} = \vec{0}$

Inner Product Space

Examples : $\mathbb{R}^n$

$$\vec{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \vec{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$$

$$\langle \vec{X}, \vec{Y} \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Inner Product Space

Examples : $\mathbb{C}^n$

$$\vec{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \vec{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{C}^n$$

$$\langle \vec{X}, \vec{Y} \rangle = x_1\overline{y_1} + x_2\overline{y_2} + \cdots + x_n\overline{y_n}$$

Inner Product Space

Examples : $L^2[(-\pi,\pi);\mathbb{R}]$ — real valued functions defined on $(-\pi,\pi)$

$$f, g, h : (-\pi, \pi) \longrightarrow \mathbb{R}$$

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) \, dx$$

Inner Product Space

Examples : $L^2[(-\pi,\pi);\mathbb{C}]$ — complex valued functions defined on $(-\pi,\pi)$

$$f, g, h : (-\pi, \pi) \longrightarrow \mathbb{C}$$

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$$

Inner Product Space

Length of a vector $\vec{u} \in V$:

$$\boxed{\begin{gathered} \|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle} \quad \text{or} \quad \|u\|^2 = \langle \vec{u}, \vec{u} \rangle \\ \left(\|\vec{u}\| \geq 0, \quad = 0 \Leftrightarrow \vec{u} = \vec{0} \right) \end{gathered}}$$

Examples:

(1) $\mathbb{R}^n$, $\|\vec{X}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

(2) $\mathbb{C}^2$, $\|\vec{X}\| = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$

Inner Product Space

Length of a vector $\vec{u} \in V$:

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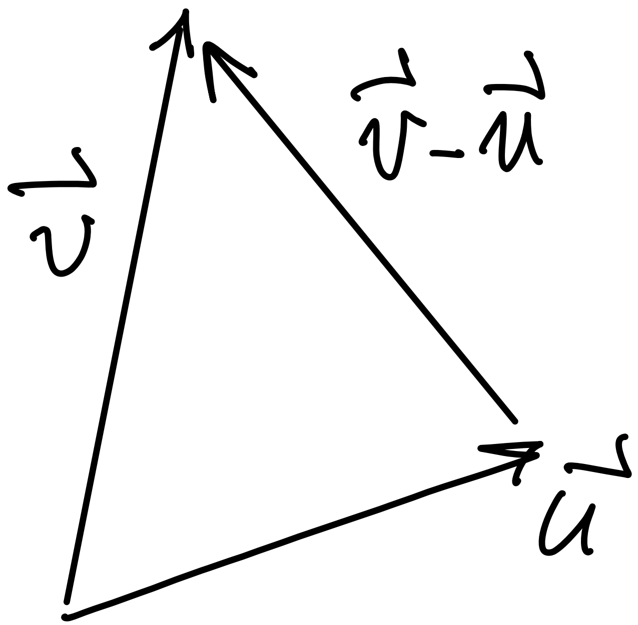
Examples:

(3) $L^2((-\pi, \pi), \mathbb{R})$: $\|f\| = \sqrt{\int_{-\pi}^{\pi} f^2 dx}$

(4) $L^2((-\pi, \pi) : \mathbb{C})$: $\|f\| = \sqrt{\int_{-\pi}^{\pi} |f|^2 dx}$

Inner Product Space

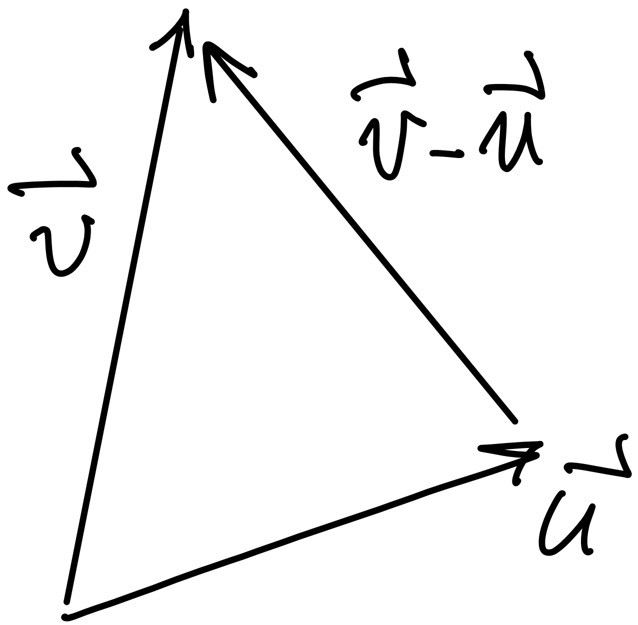
Distance between two vectors $\vec{u}, \vec{v} \in V$



$$\boxed{\begin{gathered}\|\vec{v}-\vec{u}\| = \sqrt{\langle \vec{v}-\vec{u}, \vec{v}-\vec{u} \rangle} \\ \left(\|\vec{v}-\vec{u}\|^2 = \langle \vec{v}-\vec{u}, \vec{v}-\vec{u} \rangle \right)\end{gathered}}$$

Inner Product Space

Distance between two vectors $\vec{u}, \vec{v} \in V$



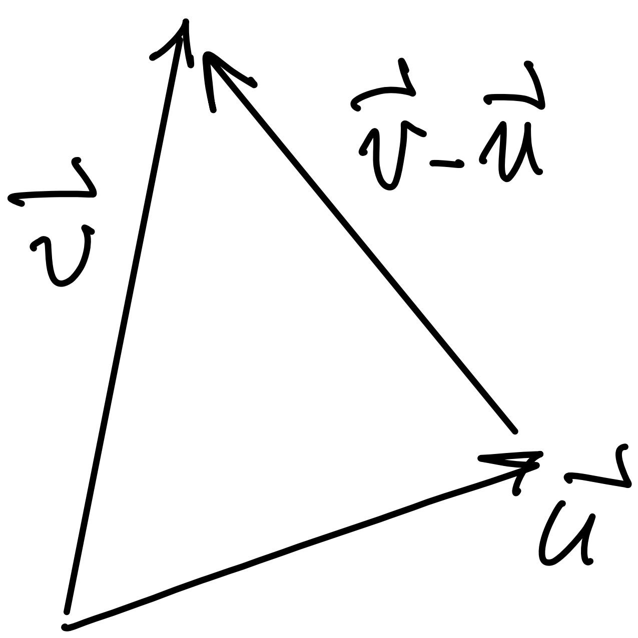
$$\boxed{\|\vec{v}-\vec{u}\| = \sqrt{\langle \vec{v}-\vec{u}, \vec{v}-\vec{u} \rangle} \\ \left(\|\vec{v}-\vec{u}\|^2 = \langle \vec{v}-\vec{u}, \vec{v}-\vec{u} \rangle \right)}$$

Example: R^n

$$\|\vec{X}-\vec{Y}\| = \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2}$$

Inner Product Space

Distance between two vectors $\vec{u}, \vec{v} \in V$



$$\|\vec{v}-\vec{u}\| = \sqrt{\langle \vec{v}-\vec{u}, \vec{v}-\vec{u} \rangle}$$

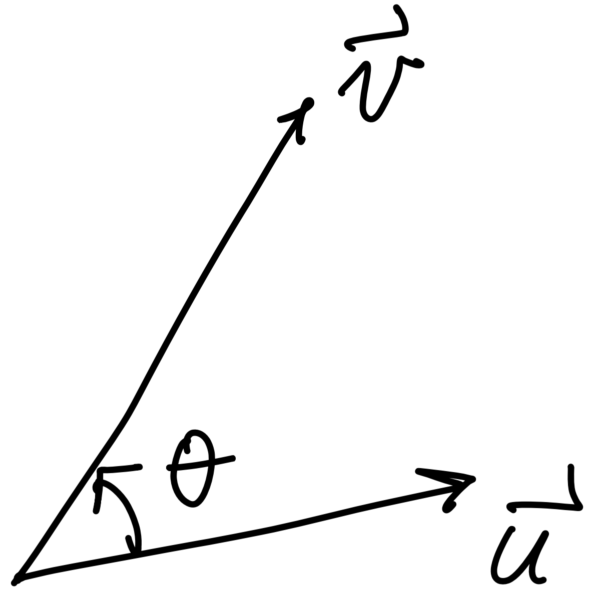
$$\left(\|\vec{v}-\vec{u}\|^2 = \langle \vec{v}-\vec{u}, \vec{v}-\vec{u} \rangle \right)$$

Example: $L^2((-\pi,\pi) : \mathbb{R})$

$$\|f-g\| = \sqrt{\int_{-\pi}^{\pi} (f-g)^2 \, dx}$$

Inner Product Space

Angle between two vectors $\vec{u}, \vec{v} \in V$



$-\pi < \theta \leq \pi$

$$\boxed{\cos\theta = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}}$$

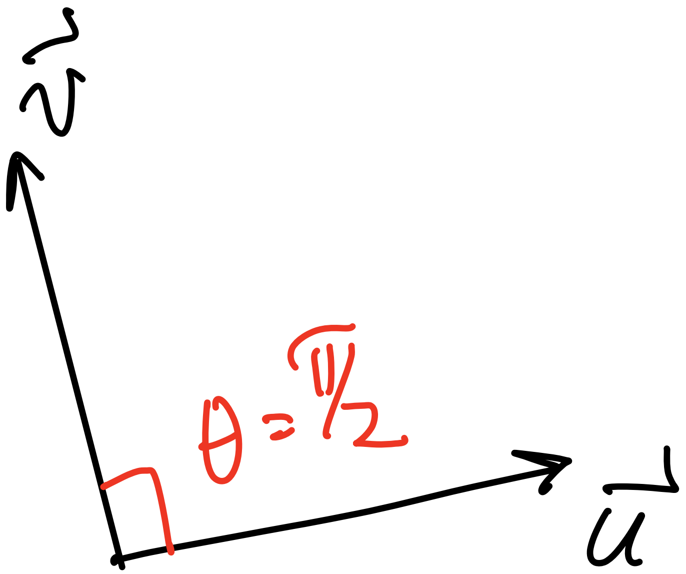
$$\boxed{\langle \vec{u}, \vec{v} \rangle = \|\vec{u}\| \|\vec{v}\| \cos\theta}$$

$$\left(\theta = \cos^{-1}\left(\frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|} \right) \right)$$

Inner Product Space

Angle between two vectors $\vec{u}, \vec{v} \in V$

$$\boxed{\langle \vec{u}, \vec{v} \rangle = \|\vec{u}\| \|\vec{v}\| \cos\theta}$$



$-\pi < \theta \leq \pi$

$$\vec{u} \perp \vec{v} \quad \text{orthogonal (perpendicular)}$$

$$\Updownarrow$$

$$\langle \vec{u}, \vec{v} \rangle = 0 \quad (\cos\theta = 0)$$

Inner Product Space

In order for $\cos\theta$ to be well-defined, we need:

$$|\cos\theta| \leqslant 1$$

i.e. $$\left|\frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\| \|\vec{v}\|}\right| \leqslant 1$$

i.e. $$|\langle \vec{u}, \vec{v} \rangle| \leqslant \|\vec{u}\| \|\vec{v}\|$$

Inner Product Space

In order for $\cos\theta$ to be well-defined, we need:

Cauchy-Schwarz Inequality:

$$|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\|\|\vec{v}\|$$

Inner Product Space

In order for $\cos\theta$ to be well-defined, we need:

Cauchy-Schwarz Inequality:

$$|\langle \vec{u}, \vec{v} \rangle| \leqslant \|\vec{u}\| \|\vec{v}\|$$

$$|\langle \vec{u}, \vec{v} \rangle| = \|\vec{u}\| \|\vec{v}\| \Leftrightarrow \vec{u} \mathbin{/\!/} \vec{v}, \text{ i.e. } \vec{u} = \alpha \vec{v} \text{ or } \vec{v} = \alpha \vec{u}$$