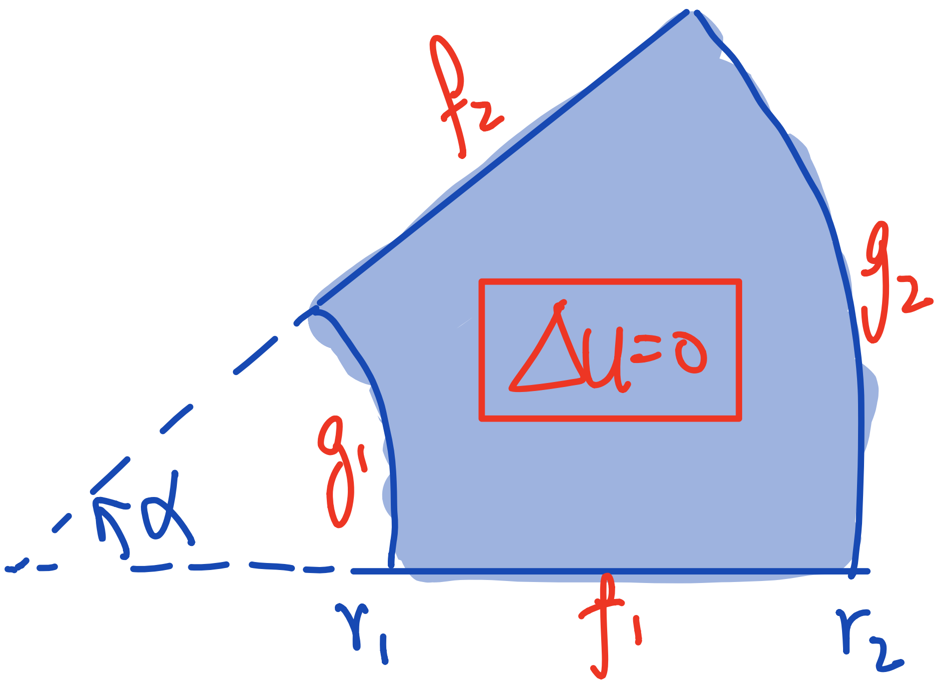
Laplace Equation in Circular Sector



Ω: in polar coord.:

$r_1 < r < r_2$

$0 < \theta < \alpha$

$$\boxed{\begin{aligned} u &= u(r, \theta) \\ \Delta u &= \left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\right) u \end{aligned}}$$

Laplace Equation in Circular Sector

$$\boxed{\begin{gathered} u = u(r,\theta) \\ \Delta u = \left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\right)u \end{gathered}}$$

Separation of Variables

$$u(r,\theta) = R(r)\,\Phi(\theta)$$

$$\boxed{\Delta u = \left(R''(r)\,\Phi(\theta) + \frac{1}{r}R'(r)\,\Phi(\theta) + \frac{1}{r^2}R(r)\,\Phi''(\theta)\right)}$$

Laplace Equation in Circular Sector

$$\boxed{\Delta u = \left(R''(r)\Phi(\theta) + \frac{1}{r}R'(r)\Phi(\theta) + \frac{1}{r^2}R(r)\Phi''(\theta) \right)} = 0$$

$$\frac{r^2 R''(r) + rR'(r)}{R(r)} = -\frac{\Phi''(\theta)}{\Phi(\theta)} = C \text{ (constant)}$$

① $\Phi''(\theta) + C\Phi(\theta) = 0$

B.C. $\Longrightarrow$ $C \geq 0$ $\Longrightarrow$ $C = \nu^2$

Laplace Equation in Circular Sector

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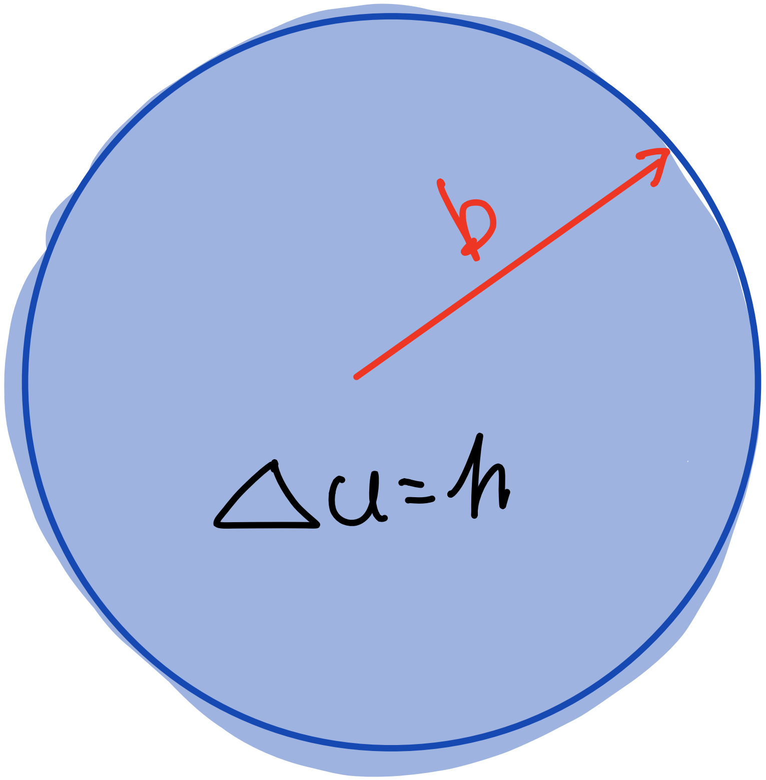
$$\frac{r^2R''(r) + rR'(r)}{R(r)} = -\frac{\Phi''(\theta)}{\Phi(\theta)} = C \text{ (constant)}$$

② $r^2R''(r) + rR'(r) - \nu^2 R(r) = 0$ (Euler Equation)

$\nu > 0$: $R(r) = Ar^{\nu} + Br^{-\nu}$

$\nu = 0$: $R(r) = A + B\log r$

Poisson Equation in a Disk



$$\Omega: \quad 0 < r < b$$
$$0 < \theta < 2\pi$$

$$\boxed{\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\right)u = h}$$

Poisson Equation in a Disk

$$\boxed{\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\right)u = h}$$

Eigenfunction expansion

(1) Let φ_n : $\underline{\Delta \varphi_n = \lambda_n \varphi_n}$

(2) Write : $h = \sum a_n \varphi_n$, $u = \sum c_n \varphi_n$

(3) $\Delta u = h$: $\Delta\left(\sum_n c_n \varphi_n\right) = \sum a_n \varphi_n$

$$\Rightarrow \sum_n \underline{\lambda_n c_n} \varphi_n = \sum \underline{a_n} \varphi_n$$

Poisson Equation in a Disk

$$\boxed{\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\right) u = h}$$

Eigenfunction expansion

(1) Let φ_n : $\Delta \varphi_n = \lambda_n \varphi_n$

(2) Write : $h = \sum a_n \varphi_n$, $u = \sum c_n \varphi_n$

(3) $\Delta u = h$: $c_n = \dfrac{a_n}{\lambda_n}$

$$\Longrightarrow \boxed{u = \sum_n \frac{a_n}{\lambda_n} \varphi_n}$$

Eigenfunction of Laplacian in a Disk

$$\boxed{\Delta u = \lambda u} \qquad (\lambda = -\mu^2 \leq 0)$$

$u = R(r)\Phi(\theta)$:

$$R''(r)\Phi(\theta) + \frac{1}{r}R'(r)\Phi(\theta) + \frac{1}{r^2}R(\theta)\Phi''(\theta) = -\mu^2 R(r)\Phi(\theta)$$

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$$\boxed{\frac{r^2 R''(r) + rR'(r)}{R(r)} + \mu^2 r^2 = -\frac{\Phi''(\theta)}{\Phi(\theta)}}$$

Eigenfunction of Laplacian in a Disk

$$\boxed{\frac{r^2R''(r)+rR'(r)}{R(r)}+\underline{\mu^2 r^2}=-\frac{\Phi''(\theta)}{\Phi(\theta)}}=C\ \text{(constant)}$$

① $\Phi''(\theta)+c\,\Phi(\theta)=0$ $\quad(\underbrace{\Phi(\theta+2\pi)=\Phi(\theta)}_{2\pi\text{-periodic}})$

$\Longrightarrow$ $\underline{C=n^2}$, $n=0,1,2,\cdots$

$\Longrightarrow$ $\Phi(\theta)=\underbrace{1}_{n=0}$, $\underbrace{\cos n\theta,\ \sin n\theta}_{n=1,2,3,\cdots}$

Eigenfunction of Laplacian in a Disk

$$\boxed{\frac{r^2R''(r)+rR'(r)}{R(r)}+\underline{\mu^2r^2}=-\frac{\Phi''(\theta)}{\Phi(\theta)}}=C$$

(constant)

② $$\frac{r^2R''(r)+rR'(r)}{R(r)}+\mu^2r^2=\nu^2$$

(e.g. $C=n^2$)

$$\boxed{r^2R''(r)+rR'(r)+(\mu^2r^2-\nu^2)R(r)=0}$$

Eigenfunction of Laplacian in a Disk

$$\boxed{\frac{r^2R''(r)+rR'(r)}{R(r)}+\underline{\mu^2r^2}=-\frac{\Phi''(\theta)}{\Phi(\theta)}}=C$$

(constant)

② $$\frac{r^2R''(r)+rR'(r)}{R(r)}+\mu^2r^2=\nu^2$$

Bessel eqn.

$\Delta u=\lambda u$

$$\boxed{r^2R''(r)+rR'(r)+(\mu^2r^2-\nu^2)R(r)=0}$$

Euler Eqn.

(Compare: $\boxed{r^2R''(r)+rR'(r)-\nu^2R(r)=0}$)

$\Delta u=0$
$(\lambda=0)$