Bessel Functions

Bessel Equation

$$\boxed{x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0}$$

$$f(x) = \sum_{k=0}^{\infty} a_k x^{k+b} \qquad (a_0 \neq 0)$$

$$= x^b \left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \right]$$

$$(\nu \neq -n) \qquad = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+\nu}}{2^{2k} k! (1+\nu)(2+\nu)\cdots(k+\nu)}$$

Bessel Functions

Bessel Equation

$$\boxed{x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0}$$

for $\nu \neq -n$ [F. p.129]

$$J_\nu(x) = \underbrace{\frac{1}{2^\nu \Gamma(\nu+1)}}_{a_0} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+\nu}}{2^{2k} k! (1+\nu)(2+\nu)\cdots(k+\nu)}$$

for $\nu \neq n$

$$J_{-\nu}(x) = \underbrace{\frac{1}{2^{-\nu} \Gamma(-\nu+1)}}_{a_0} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k-\nu}}{2^{2k} k! (1-\nu)(2-\nu)\cdots(k-\nu)}$$

Bessel Functions

Bessel Equation

$$\boxed{x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0}$$

[F. p. 131]
(5.10)

$$\boxed{J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}}$$

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,dt \quad \text{— Gamma function}$$

(for $\mathrm{Re}\,z > 0$)

$\Gamma(z+1) = z\,\Gamma(z)$, $\quad \Gamma(k+\nu+1) = \Gamma(\nu+1)(1+\nu)(2+\nu)\cdots(k+\nu)$

$\Gamma(n+1) = n!$,

Bessel Function of First Kind

$$\boxed{J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}}$$

When $\nu^2 \neq n^2$ (ie. $\nu \neq n$ or $\nu \neq -n$),
$J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent
as $J_\nu(x)$ & $J_{-\nu}(x)$ has different behaviors
as $x \to 0$:

$$J_\nu(x) \sim \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu}$$

$$J_{-\nu}(x) \sim \frac{1}{\Gamma(-\nu+1)}\left(\frac{x}{2}\right)^{-\nu}$$

Bessel Function of First Kind

$$\boxed{J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}}$$

When $\nu^2 = n^2$ (i.e. $\nu = n$ or $\nu = -n$)

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} \left(\frac{x}{2}\right)^{2k+n}$$

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+n}}{k\,(k+n)!} \left(\frac{x}{2}\right)^{2k+n}$$

Bessel Function of First Kind

$$\boxed{J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}}$$

When $\nu^2 = n^2$ (i.e. $\nu = n$ or $\nu = -n$)

Note: $J_{-n}(x) = (-1)^n J_n(x)$

i.e. $J_{-n}(x)$ and $J_n(x)$ are linearly dependent.

Bessel Function of First Kind

$$\boxed{J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}}$$

When $\nu^2 = n^2$ (i.e. $\nu = n$ or $\nu = -n$)

Define:

$$Y_\nu(x) = \frac{(\cos \nu\pi)\, J_\nu(x) - J_{-\nu}(x)}{(\sin \nu\pi)}$$

Bessel Function of Second Kind

$$\boxed{Y_\nu(x) = \frac{(\cos \nu\pi) J_\nu(x) - J_{-\nu}(x)}{(\sin \nu\pi)}}$$

$$\left(Y_n(x) = \frac{0}{0} \right)$$

Define :

$$Y_n(x) = \lim_{\nu \to n} \frac{(\cos \nu\pi) J_\nu(x) - J_{-\nu}(x)}{(\sin \nu\pi)}$$

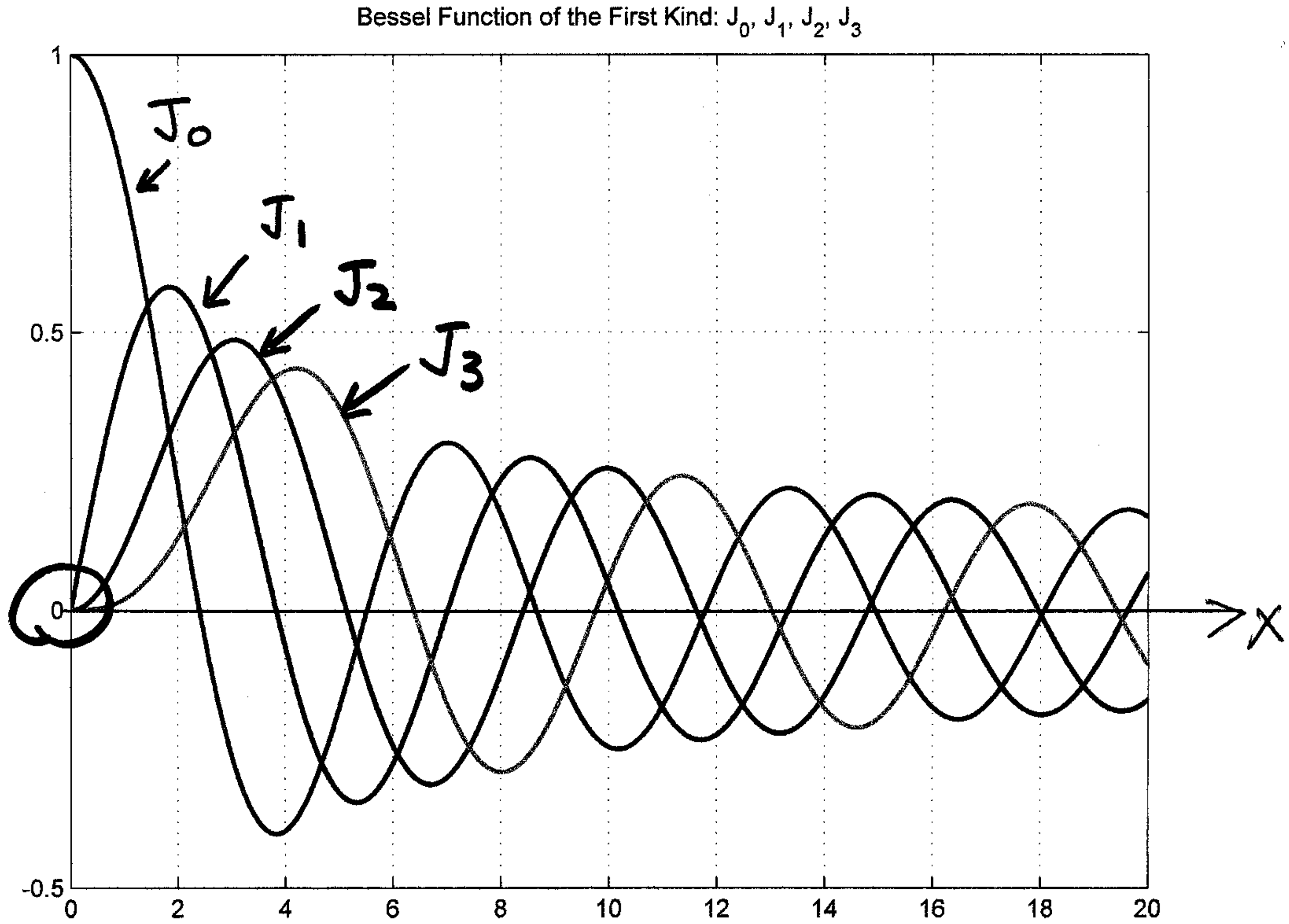
Bessel Function of Second Kind

$J_n(x)$ and $Y_n(x)$ are linearly independent.

$$J_n(x) \underset{x\to 0^+}{\cong} \frac{1}{n!}\left(\frac{x}{2}\right)^n \qquad n=0,1,2,3,\cdots$$

$$Y_n(x) \underset{x\to 0^+}{\cong} \begin{cases} -\dfrac{(n-1)!}{\pi}\left(\dfrac{x}{2}\right)^{-n} \approx -\infty & n=1,2,3,\cdots \\ \dfrac{2}{\pi}\log\left(\dfrac{x}{2}\right) \approx -\infty & n=0 \end{cases}$$

Bessel Function of First Kind



Student Version of MATLAB

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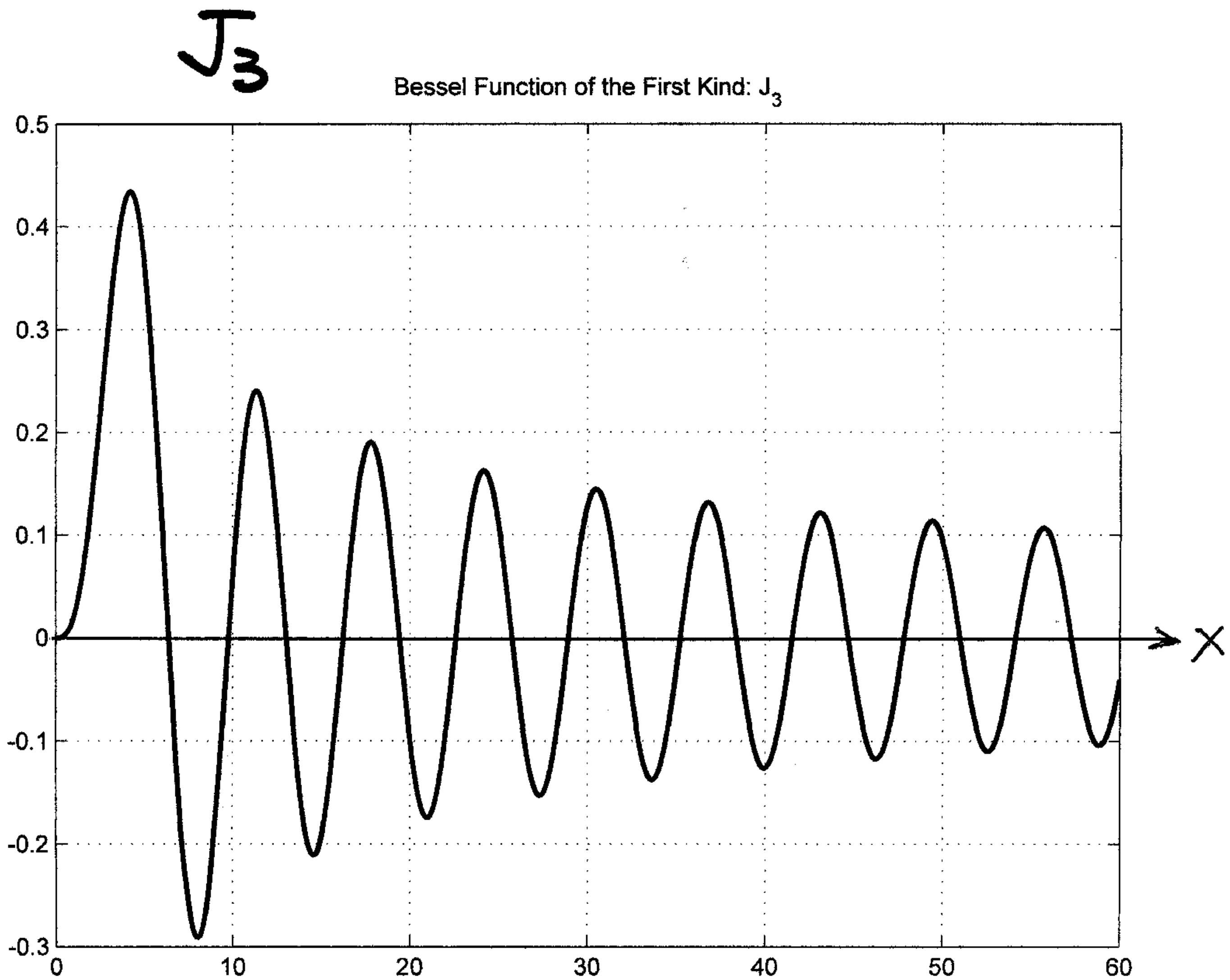

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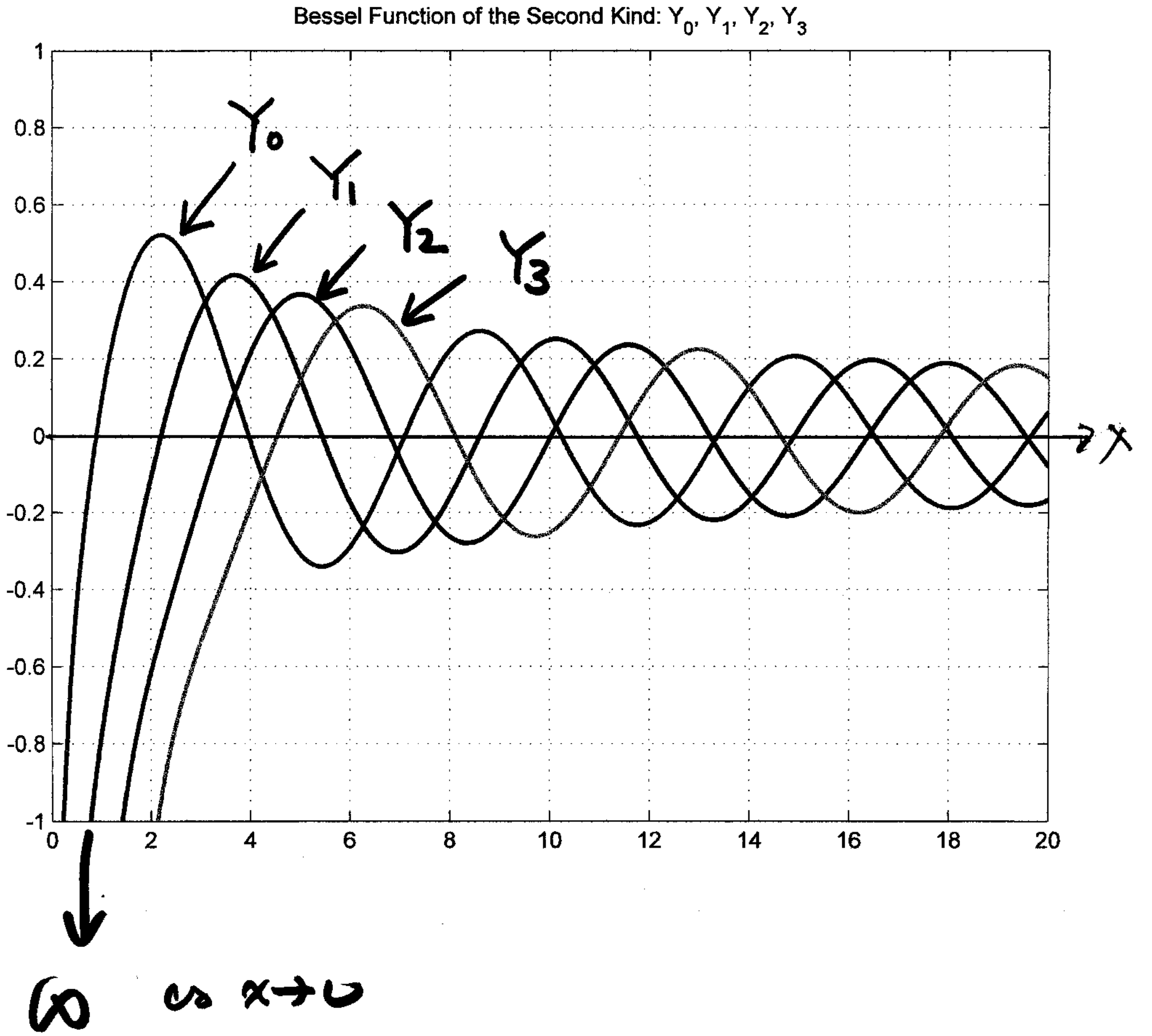
Bessel Function of the Second Kind



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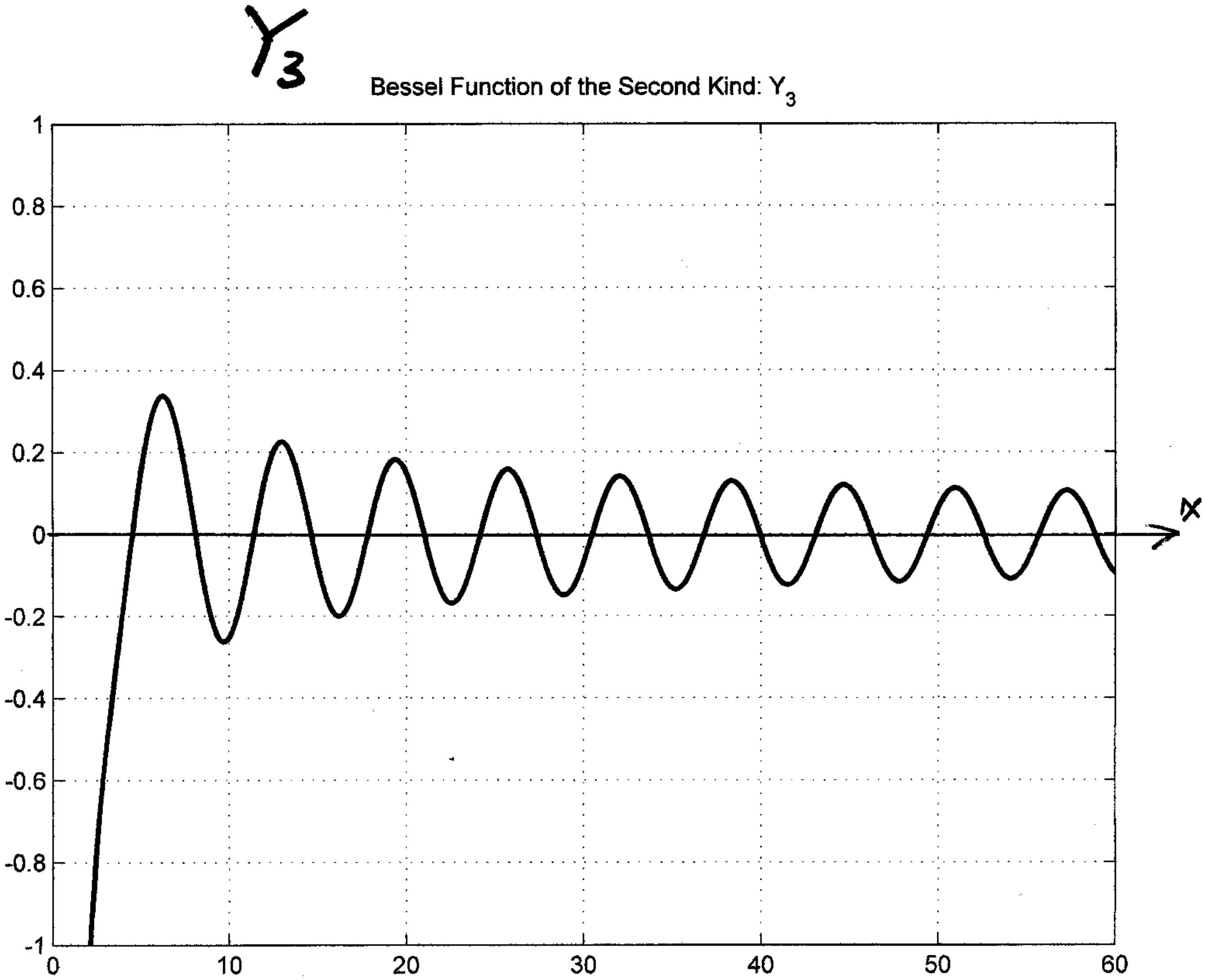

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Conclusion

$$(*)_p \qquad x^2 f''(x) + x f'(x) + (x^2 - p^2) f(x) = 0, \quad x > 0$$

The solutions are given by:

If $p^2 \neq n^2$; $\quad f(x) = A J_p(x) + B J_{-p}(x)$

If $p^2 = n^2$; $\quad f(x) = A J_n(x) + B Y_n(x)$

$J_p(x)$ = Bessel function of 1^{st} kind

$Y_n(x)$ = Bessel function of 2^{nd} kind.

Bessel Functions: Properties & Asymptotics

$$\boxed{x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0}$$

Recurrence Formulas [F. p. 133]

$$\frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] = -x^{-\nu} J_{\nu+1}(x)$$

$$\frac{d}{dx}\left[x^{\nu} J_\nu(x)\right] = x^{\nu} J_{\nu-1}(x)$$

$$x J_\nu'(x) - \nu J_\nu(x) = -x J_{\nu+1}(x)$$

$$x J_\nu'(x) + \nu J_\nu(x) = x J_{\nu-1}(x)$$

$$x J_{\nu-1}(x) + x J_{\nu+1}(x) = 2\nu J_\nu(x)$$

$$J_{\nu-1}(x) - J_{\nu+1}(x) = 2 J_\nu'(x)$$

Bessel Functions: Properties & Asymptotics

$$\boxed{x^2 f(x) + x f(x) + (x^2 - \nu^2) f(x) = 0}$$

$\nu = n + \frac{1}{2}$ [F. p. 134]

(1) $J_{-\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x, \quad J_{\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x$

(2) $J_{3/2}(x) = x^{-1} J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \left(\frac{\sin x}{x} - \cos x\right)$

(3) $J_\nu(x) = x^{-\frac{1}{2}} \left[P_\nu(x) \cos x + Q_\nu(x) \sin x \right]$

rational functions

Generating Function (z-transform)

$$\boxed{\sum_{n=-\infty}^{\infty} J_n(x)\, z^n = \exp\left[\frac{x}{2}\left(z - \frac{1}{z}\right)\right]}$$

[F. p.134 ~ p.136]

$$z = e^{i\theta} \Longrightarrow e^{ix\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(x)\, e^{in\theta}$$

$$\overset{\text{(F.S.)}}{\Longrightarrow} \boxed{J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin\theta}\, e^{-in\theta}\, d\theta}$$

Generating Function (z-transform)

$$\boxed{\sum_{n=-\infty}^{\infty} J_n(x) z^n = \exp\left[\frac{x}{2}\left(z-\frac{1}{z}\right)\right]}$$

[F. p.134 – p.136]

Bessel's Integral Formula

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta - n\theta)\, d\theta$$

$$J_n(x) = \frac{2}{\pi}\int_0^{\pi} \cos(x\sin\theta)\cos n\theta\, d\theta \quad (n = \text{even})$$

$$J_n(x) = \frac{2}{\pi}\int_0^{\pi} \sin(x\sin\theta)\sin n\theta\, d\theta \quad (n = \text{odd})$$

Zeros of Bessel Functions

Thm 5.1 p. 139 for $x \geq 1$:

$$\boxed{J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + \text{error}(x)}$$

$$\boxed{J_{-\nu}(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x + \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + \text{error}(x)}$$

$\sim \frac{1}{x^{1/2}}$ $\qquad$ $\sim \frac{1}{x^{3/2}}$

$$\boxed{Y_\nu(x) \simeq \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)}$$