Bessel Function — Asymptotics

$$\boxed{x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0}$$

[F. p 134] $\nu = n + \frac{1}{2}, \quad n = 0, \pm 1, \pm 2, \cdots$

(1) $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x, \quad J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

(2) $J_{\frac{3}{2}}(x) = x^{-1} J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$

(3) $J_{\nu}(x) = \frac{1}{\sqrt{x}} \left[P_{\nu}(x) \cos x + Q_{\nu}(x) \sin x \right]$

rational functions

Bessel Function — Asymptotics

$$\boxed{x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0}$$

$$f(x) = \frac{1}{\sqrt{x}} g(x)$$

$$\boxed{g''(x) + g(x) + \frac{\left(\frac{1}{4} - \nu^2\right)}{x^2} g(x) = 0}$$

$$g(x) \overset{(x \gg 1)}{\approx} A \cos x + B \sin x = C \cos(x - \varphi)$$

$$\boxed{f(x) \overset{(x \gg 1)}{\approx} \frac{1}{\sqrt{x}} C_\nu \cos(x - \varphi_\nu)}$$

Bessel Function — Asymptotics

$$\boxed{x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0}$$

Thm 5.1 p.139 for $x \geq 1$:

$$\boxed{J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + \text{error}(x)}$$

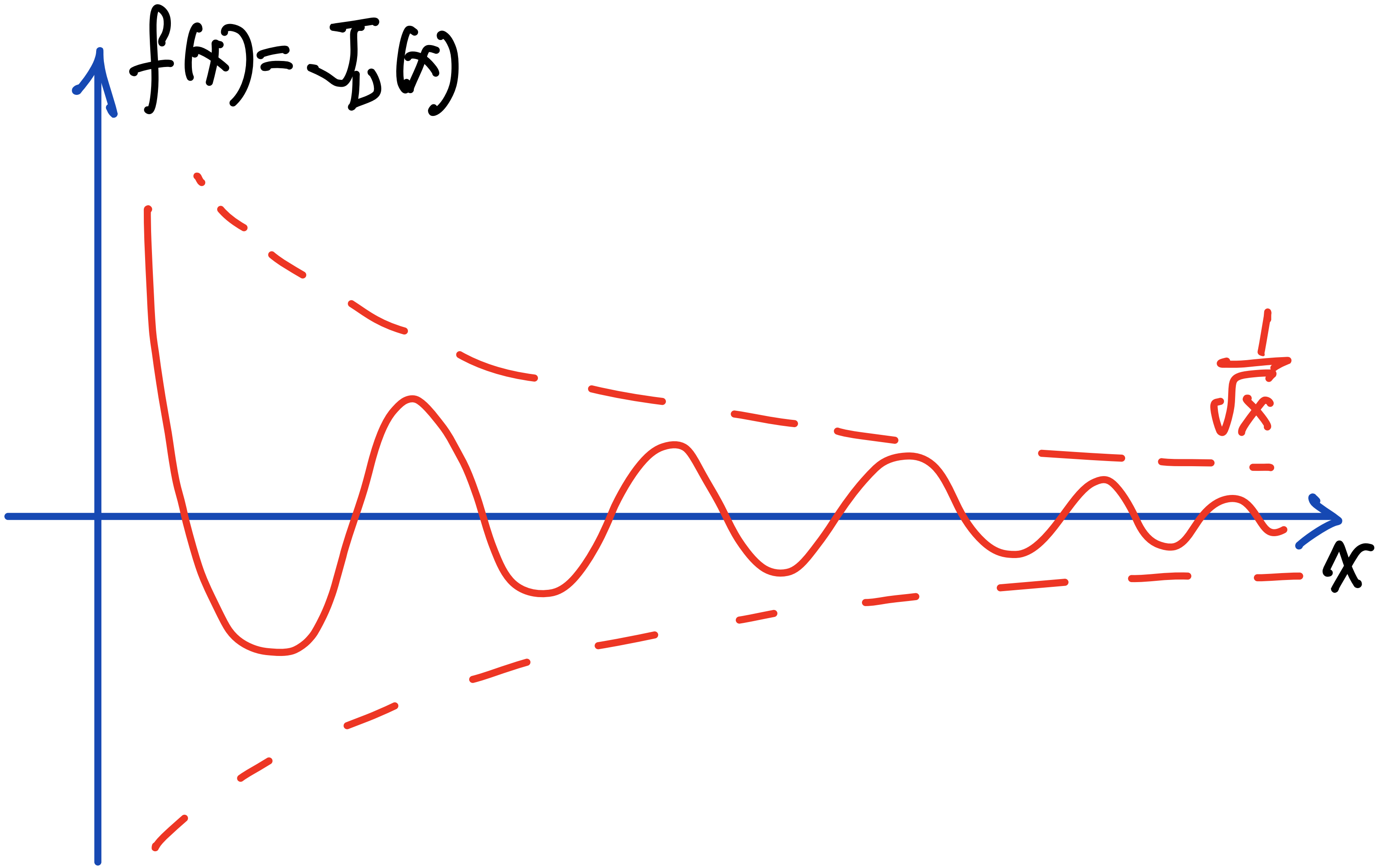
$$\boxed{J_{-\nu}(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x + \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + \text{error}(x)}$$

$\sim \frac{1}{x^{1/2}}$ $\sim \frac{1}{x^{3/2}}$

$$\boxed{Y_\nu(x) \simeq \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)} \quad (\nu \neq n)$$

Bessel Function — Asymptotics

$$\boxed{x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0}$$



Bessel Function — Zeros

Thm 5.2 $\quad a, b \geq 0$, $(a,b) \neq (0,0)$, $\{\lambda_k\}_{k\geq 1}$ zeros of $a J_\nu(x) + b x J_\nu'(x)$

(1) $(b=0)$ $\quad \boxed{J_\nu(\lambda_k) = 0}$

$$\left(\cos\left(\lambda_k - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \approx 0 \right)$$

$$\boxed{\lambda_k \sim \left(k + M(\nu) + \frac{\nu}{2} + \frac{3}{4}\right)\pi, \quad k \gg 1}$$

Bessel Function — Zeros

Thm 5.2 $a, b \geq 0$, $(a,b) \neq (0,0)$, $\{\lambda_k\}_{k\geq 1}$ zeros of $a J_\nu(x) + b x J_\nu'(x)$

(2) $(b > 0)$ $\boxed{a J_\nu(\lambda_k) + \underbrace{b \lambda_k J_\nu'(\lambda_k)}_{\text{Dominating term as } \lambda_k \gg 1} = 0}$

$$\left(\sin\left(\lambda_k - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \approx 0 \right)$$

$$\boxed{\lambda_k \sim \left(k + M(\nu) + \frac{\nu}{2} + \frac{1}{4}\right)\pi, \quad k \gg 1}$$

Bessel Equation and Sturm-Liouvill Op.

$$\boxed{r^2R''(r)+rR'(r)+(\mu^2r^2-\nu^2)R(r)=0}$$

$$(a\leq r\leq b)$$

$$\Downarrow$$

$$\underbrace{(rR'(r))'-\frac{\nu^2}{r}R(r)}_{LR}=-\mu^2 rR(r)$$

eigenvalue

weight, $w(r)=r$

$$\boxed{LR(r)=(-\mu^2)\,rR(r)}$$

Bessel Equation and Sturm-Liouvill Op.

$$\boxed{r^2 R''(r) + r R'(r) + (\mu^2 r^2 - \nu^2) R(r) = 0}$$

$$(a \leq r \leq b)$$

B.C. $$\boxed{\begin{aligned} \alpha R(a) + \alpha' R'(a) &= 0 \\ \beta R(b) + \beta' R'(b) &= 0 \end{aligned}}$$

If $a = 0$, then

$$\boxed{R(0^+) \text{ exists, finite, } \beta R(b) + \beta' R'(b) = 0}$$

Bessel Equation and Sturm-Liouvill Op.

$$LR = (rR')' - \frac{\nu^2}{r} R, \quad a \le r \le b$$

with B.C.

$$\boxed{\text{Thm} \quad L \text{ is self-adjoint} \quad \text{i.e. } \langle LR, S \rangle = \langle R, LS \rangle}$$

$$\left(\int_a^b \left((rR')' - \frac{\nu^2}{r} R \right) S \, dr = \int_a^b R \left((rS')' - \frac{\nu^2}{r} S \right) dr \right)$$

Bessel Equation and Sturm-Liouvill Op.

$$LR = (rR')' - \frac{\nu^2}{r}R, \quad a \le r \le b$$

with B.C.

Thm $\quad LR = -\mu_1^2 rR \quad a \le r \le b$

$\quad LS = -\mu_2^2 rS \quad$ with B.C.

Then $\quad R \perp S$ in $L^2_{w(r)}$ weighted L^2, $w(r) = r^2$

i.e. $\quad \displaystyle\int_a^b R(r)S(r)\,r\,dr = 0$

Eigenvalues and Zeros of Bessel Func's

$$r^2 R'' + rR' + (\mu^2 r^2 - \nu^2)R = 0, \quad 0 \leq a \leq r \leq b$$

$$\Updownarrow$$

$$(rR')' - \frac{\nu^2}{r} R = -\mu^2 r R, \quad 0 \leq a \leq r \leq b$$

$-\mu^2$ = eigenvalue

B.C. $\begin{cases} \alpha R(a) + \alpha' R'(a) = 0 \\ \beta R(b) + \beta' R'(b) = 0 \end{cases}$

If $\underline{a = 0}$, then $\begin{cases} R(0^+) \text{ exists and is finite} \\ \beta R(b) + \beta' R'(b) = 0 \end{cases}$

Eigenvalues and Zeros of Bessel Func's

$$r^2R'' + rR' + (\mu^2 r^2 - \nu^2)R = 0, \quad 0 \le a \le r \le b$$

$$(rR')' - \frac{\nu^2}{r}R = -\mu^2 rR, \quad 0 \le a \le r \le b$$

$-\mu^2$ = eigenvalue

$$\boxed{R(r) = A\,J_\nu(\mu r) + B\,Y_\nu(\mu r)}$$

If $a = 0$ $\Rightarrow$ $R(0^+)$ exists and is finite

$\Rightarrow$ $B = 0$

$\Rightarrow$ $\boxed{R(r) = A\,J_\nu(\mu r)}$ $\quad 0 \le r \le b$

Eigenvalues and Zeros of Bessel Func's

$$\boxed{R(r) = A J_\nu(\mu r)} \qquad 0 \leq r \leq b$$

$$\beta R(b) + \beta' R'(b) = 0$$

$$\Longleftrightarrow \quad \beta J_\nu(\mu b) + \beta' \mu J_\nu'(\mu b) = 0$$

$$\Longleftrightarrow \quad \beta J_\nu(\mu b) + \frac{\beta'}{b}(\mu b) J_\nu'(\mu b) = 0$$

$$\Longleftrightarrow \quad \boxed{\beta J_\nu(\lambda_k) + \frac{\beta'}{b} \lambda_k J_\nu'(\lambda_k) = 0}$$

Set $\boxed{\mu b = \lambda_k \Rightarrow \mu = \frac{\lambda_k}{b}}$

Eigenvalues and Zeros of Bessel Func's

$$\boxed{R(r) = A J_\nu(\mu r)} \qquad 0 \leqslant r \leqslant b$$

Set $\boxed{\mu b = \lambda_k \Rightarrow \mu = \frac{\lambda_k}{b}}$

$$\Longrightarrow \boxed{R(r) = R_k(r) = J_\nu\left(\frac{\lambda_k}{b} r\right)} \qquad 0 \leqslant r \leqslant b$$

Implicitly we assume that $\lambda_k \neq 0, \mu \neq 0$

Eigenvalues and Zeros of Bessel Func's

$$\boxed{R(r) = A J_\nu(\mu r)} \qquad 0 \leq r \leq b$$

Set $\boxed{\mu b = \lambda_k \Rightarrow \mu = \frac{\lambda_k}{b}}$

$$\Longrightarrow \boxed{R(r) = R_k(r) = J_\nu\left(\frac{\lambda_k}{b} r\right)} \qquad 0 \leq r \leq b$$

If $\mu = 0$, then R solves

$r^2 R'' + rR' - \nu^2 R = 0$, Euler Equation

Bessel Function and their Orthogonality

$a=b$

Thm 5.3 $(\nu \geq 0, \quad 0 \leq r \leq b, \quad w(r) = r)$

(1) $\boxed{\text{B.C. } \beta' = 0, \text{ i.e. } R(b) = 0}$

Let $\{\lambda_k\}_{k\geq 1}$ be the pos. zeros of $J_\nu(x)$

i.e. $J_\nu(\lambda_k) = 0$

Let $\boxed{\phi_k(r) = J_\nu\left(\frac{\lambda_k}{b} r\right)}$

Bessel Function and their Orthogonality

Thm 5.3 $\left(\nu \geq 0, \quad \overset{a=b}{0} \leq r \leq b, \quad w(r) = r\right)$

(1) $\boxed{\text{B.C. } \beta' = 0, \text{ i.e. } R(b) = 0}$

Then $\{\phi_k\}_{k\geq 1}$ is an orth. basis for $L^2_w(0,b)$

i.e.

$$\boxed{\underset{k\neq j}{\langle \phi_k, \phi_j \rangle_{L^2_w(0,b)}} = \int_0^b \phi_k(r)\,\phi_j(r)\,r\,dr = 0}$$

Bessel Function and their Orthogonality

Thm 5.3 $\left(\nu \geq 0, \quad 0 \leq r \leq b, \quad w(r) = r\right)$

$a = b$

(1) $\boxed{\text{B.C. } \beta' = 0, \text{ i.e. } R(b) = 0}$

Then $\{\phi_k\}_{k \geq 1}$ is an orth. basis for $L^2_w(0,b)$

and

$$\boxed{\|\phi_k\|^2_{L^2_w(0,b)} = \int_0^b \phi_k^2(r)\, r\, dr = \frac{b^2}{2} J_{\nu+1}(\lambda_k)}$$

Bessel Function and their Orthogonality

Thm 5.3 $\left(\nu \geq 0, \; \overset{a=b}{0} \leq r \leq b \;, \; w(r) = r\right)$

(2) $\boxed{\text{B.C. } \beta' \neq 0, \text{ i.e. } \beta R(b) + \beta' R'(b) = 0}$

Let $\{\tilde{\lambda}_k\}_{k \geq 1}$ be the pos. zeros of

$$\beta J_\nu(x) + \frac{\beta'}{b} x J_\nu'(x)$$

i.e. $\beta J_\nu(\tilde{\lambda}_k) + \frac{\beta'}{b} \tilde{\lambda}_k J_\nu'(\tilde{\lambda}_k) = 0$

$$\Longleftrightarrow \boxed{\left(\frac{\beta b}{\beta'}\right) J_\nu(\tilde{\lambda}_k) + \tilde{\lambda}_k J_\nu'(\tilde{\lambda}_k) = 0}$$

$c = \beta b / \beta'$

Bessel Function and their Orthogonality

Thm 5.3 ($\nu \geq 0$, $\overset{a=b}{\cancel{0}} \leq r \leq b$, $w(r) = r$)

(2) $\boxed{\text{B.C. } \beta' \neq 0, \text{ i.e. } \beta R(b) + \beta' R'(b) = 0}$

Let $\{\tilde{\lambda}_k\}_{k \geq 1}$ be the pos. zeros of

$$\beta J_\nu(x) + \frac{\beta'}{b} x J_\nu'(x)$$

i.e. $\beta J_\nu(\tilde{\lambda}_k) + \frac{\beta'}{b} \tilde{\lambda}_k J_\nu'(\tilde{\lambda}_k) = 0$

$\Longleftrightarrow$ $\boxed{c J_\nu(\tilde{\lambda}_k) + \tilde{\lambda}_k J_\nu'(\tilde{\lambda}_k) = 0}$

Bessel Function and their Orthogonality

a = b

Thm 5.3 $(\nu \geq 0, \; 0 \leq r \leq b \;, \; w(r) = r)$

(2) $\boxed{\text{B.C. } \beta' \neq 0, \text{ i.e. } \beta R(b) + \beta' R'(b) = 0}$

Let $\boxed{\psi_k(r) = J_\nu\left(\frac{\tilde{\lambda}_k r}{b}\right)} \quad k \geq 1$

If $c > -\nu$, then

$\{\psi_k\}_{k \geq 1}$ is an orth. basis for $L^2_w(0,b)$

Bessel Function and their Orthogonality

<u>Thm 5.3</u> $\left(\nu \geq 0,\quad 0 \leq r \leq b,\quad w(r) = r\right)$

$a = b$

(2) $\boxed{\text{B.C. } \beta' \neq 0, \text{ i.e. } \beta R(b) + \beta' R'(b) = 0}$

Let $\boxed{\psi_k(r) = J_\nu\left(\frac{\tilde{\lambda}_k r}{b}\right)}$ $k \geq 1$

If $c = -\nu$, then $\left(\psi_0(r) = r^\nu, \text{ eigenvalue} = 0\right)$

$\{\psi_k\}_{k \geq 0}$ is an orth. basis for $L^2_w(0,b)$

Bessel Function and their Orthogonality

Thm 5.3 $\left(\nu \geq 0,\quad \overset{a=b}{0} \leq r \leq b \quad , \; w(r) = r\right)$

(2) $\boxed{\text{B.C. } \beta' \neq 0, \text{ i.e. } \beta R(b) + \beta' R'(b) = 0}$

Let $\boxed{\psi_k(r) = J_\nu\left(\frac{\tilde{\lambda}_k r}{b}\right)}$ $\quad k \geq 1$

$$\boxed{\underset{k \geq 1}{\|\psi_k\|_w^2} = \frac{b^2\left(\tilde{\lambda}_k^2 - \nu^2 + c^2\right)}{2\tilde{\lambda}_k^2} J_\nu(\tilde{\lambda}_k)^2, \quad \|\psi_0\|_w^2 = \frac{b^{2\nu+2}}{2\nu+2}}$$

Lemma 5.3

$$\boxed{\begin{gathered}\text{Consider} \quad r^2R'' + rR' + (\mu^2r^2 - \nu^2)R = 0 \\ \left((rR')' - \frac{\nu^2}{r}R = -\mu^2 rR \right) \\ \nu \geq 0, \quad R(0^+) \text{ exists and is finite,} \\ \beta R(b) + \beta' R'(b) = 0\end{gathered}}$$

(1) $-\mu^2 = 0$ is an eigenvalue

(i.e. there is a non-trivial solution R)

$\Longleftrightarrow$ $c = -\nu$ (i.e. $\frac{\beta}{\beta'} = -\frac{\nu}{b}$)

Lemma 5.3

Consider $r^2R'' + rR' + (\mu^2 r^2 - \nu^2)R = 0$

$$\left((rR')' - \frac{\nu^2}{r} R = -\mu^2 r R \right)$$

$\nu \geq 0$, $R(0^+)$ exists and is finite,

$$\beta R(b) + \beta' R'(b) = 0$$

(2) If $\beta' = 0$ or $c > -\nu$, i.e. $\frac{\beta}{\beta'} > -\frac{\nu}{b}$

then there is no negative eigenvalue.