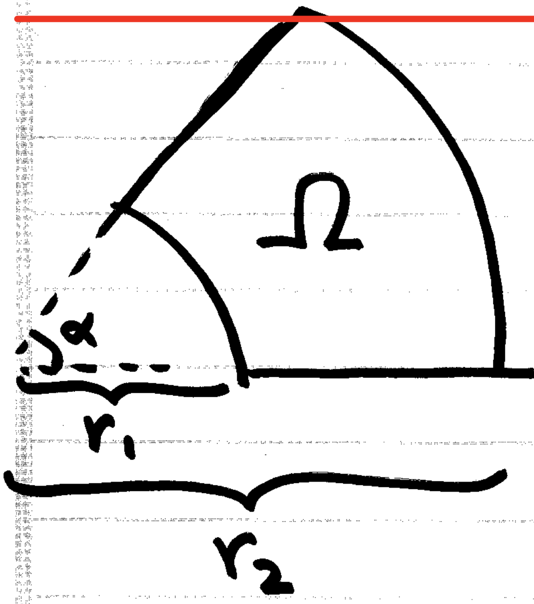
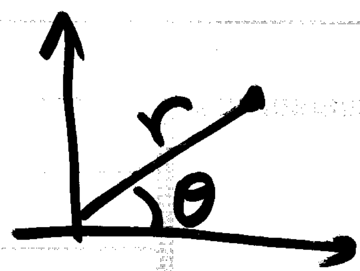
Dirichlet Problem in a Circular Sector.

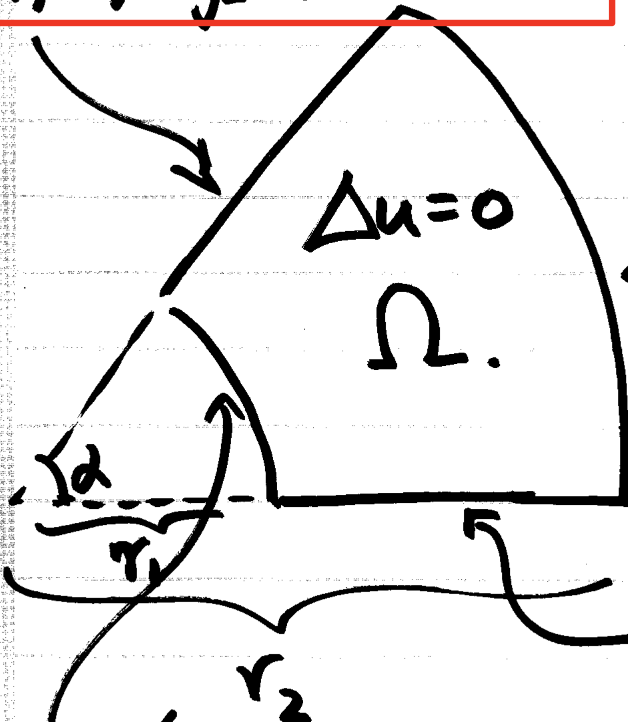


(Polar Coord.)

$$\Omega = \{ (r_1 \leq r \leq r_2); \; (0 \leq \theta \leq \alpha) \}$$



$$\boxed{u(r,\alpha)=g_2(r) \quad r_1<r<r_2}$$



$$\boxed{u(r_2,\theta)=f_2(\theta)}$$

$0 \le \theta \le \alpha$

$$\boxed{u(r,0)=g_1(r)}$$

$r_1 \le r \le r_2$

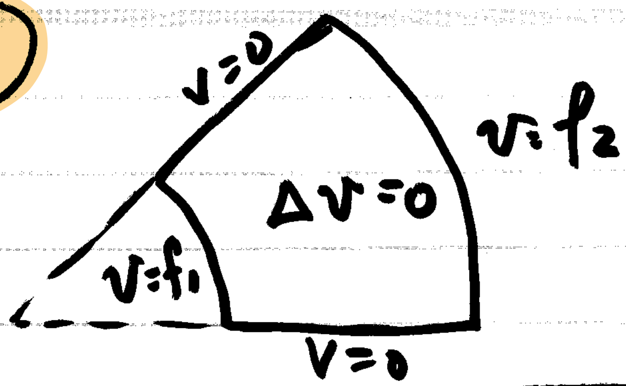
$$\boxed{u(r_1,\theta)=f_1(\theta)}$$

$0 \le \theta \le \alpha$

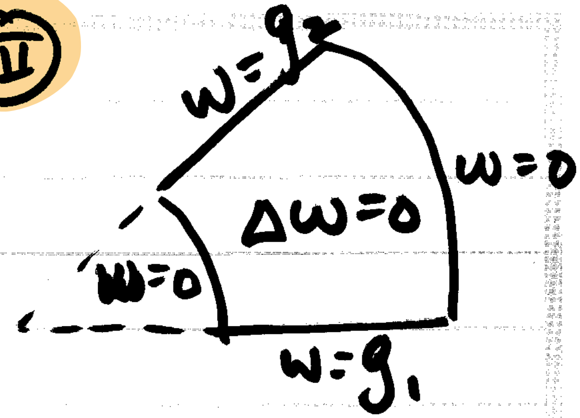
I

II

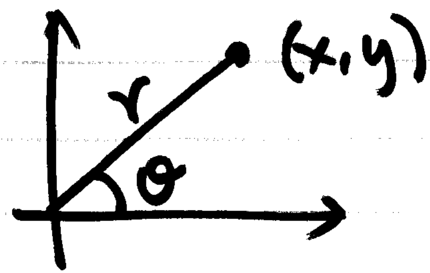
I



II



$$\boxed{u = v + w}$$



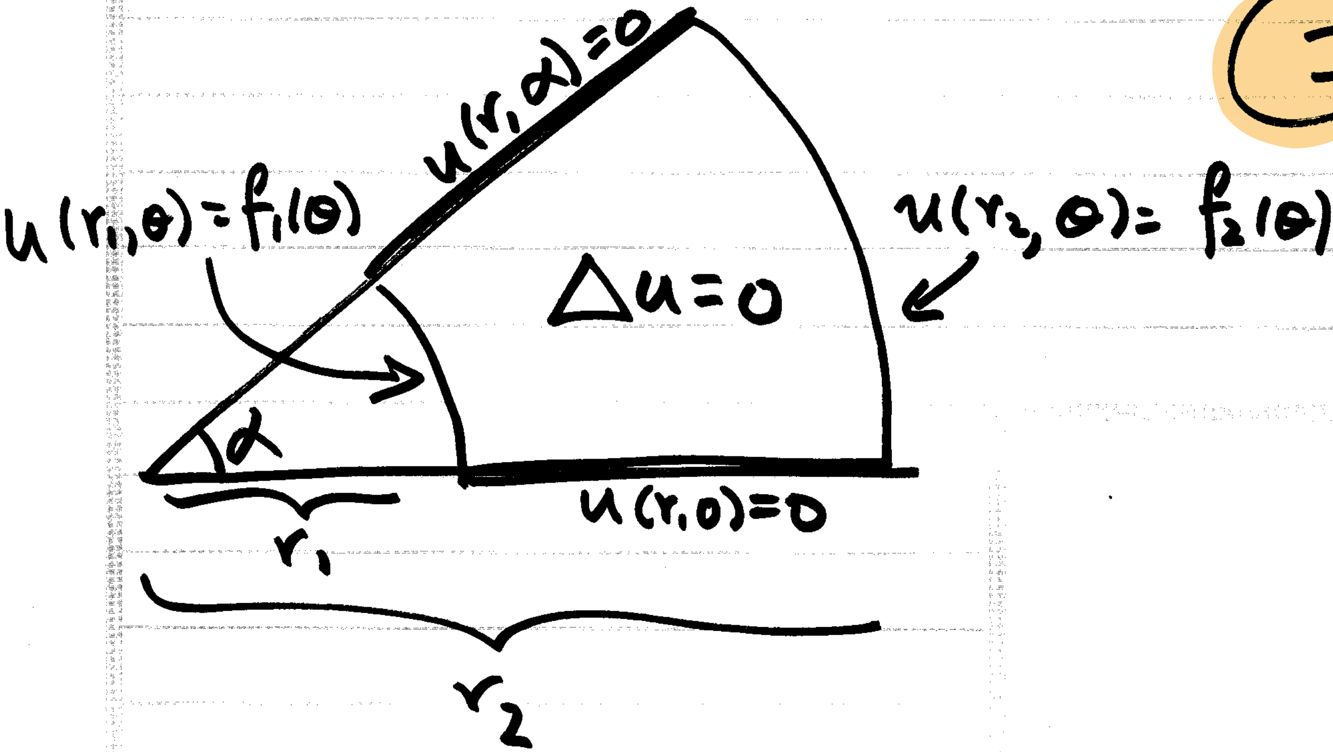
$$x = r\cos\theta, \quad y = r\sin\theta \quad \longleftrightarrow \quad r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}\frac{y}{x}$$

$$\Delta v = v_{xx}(x,y) + v_{yy}(x,y) \quad \leftarrow \quad (x,y)$$

$$= \boxed{v_{rr}(r,\theta) + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta}} \quad (r,\theta)$$

$$\Delta = \underbrace{\boxed{\partial_x^2 + \partial_y^2}}_{(x,y).} = \underbrace{\boxed{\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2}}_{(r,\theta)}$$

I



Guess (by separation of variables)

$$u(r,\theta) = R(r)\,\Phi(\theta) \qquad r_1 \le r \le r_2$$

$$0 \le \theta \le \alpha$$

$$[R\Phi]_{rr} + \frac{1}{r}[R\Phi]_r + \frac{1}{r^2}[R\Phi]_{\theta\theta} = 0$$

$$R''\Phi + \frac{1}{r}R'\Phi + \frac{1}{r^2}R\Phi'' = 0 \quad \times \frac{1}{R\Phi}$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Phi''}{\Phi} = 0$$

$$\underbrace{\underbrace{\frac{r^2 R'' + rR'}{R}}_{r} = \underbrace{-\frac{\Phi''}{\Phi}}_{\theta}} = \nu \quad (\text{a const})$$

$$\begin{cases} \Phi'' = -\nu \Phi & \leftarrow \text{B.C.?} \quad \Phi(0) = \Phi(\alpha) = 0 \\ r^2 R'' + rR' - \nu R = 0 \quad ? \end{cases}$$

$\nu > 0$

1-D, Dir, $\boxed{\nu = \frac{n^2\pi^2}{\alpha^2}, \quad \Phi(\theta) = \sin\frac{n\pi\theta}{\alpha}.}$

$$r^2 R'' + r R' - \nu R = 0$$

2nd order, linear, non-constant coeff ODE

Euler Equation

$\lambda = ?$

Guess: $R(r) = r^\lambda$

$$r^2 [r^\lambda]'' + r[r^\lambda]' - \nu [r^\lambda] = 0$$

$$r^2 (\lambda)(\lambda-1) r^{\lambda-2} + r \lambda (r^{\lambda-1}) - \nu r^\lambda = 0$$

$$\lambda(\lambda-1) \cancel{r^\lambda} + \lambda \cancel{r^\lambda} - \nu \cancel{r^\lambda} = 0$$

$$\lambda(\lambda-1) + \lambda - \nu = 0 \Longrightarrow \text{Solve for } \lambda$$

$$\lambda^2 - \cancel{\lambda} + \cancel{\lambda} - \nu = 0$$

$$\lambda^2 = \nu \quad \infty \quad \lambda^2 = \frac{n^2\pi^2}{\alpha^2},$$

$$\lambda = \pm \frac{n\pi}{\alpha}, \quad n \geq 1$$

$$\boxed{R(r) = A r^{\frac{n\pi}{\alpha}} + B r^{-\frac{n\pi}{\alpha}}}$$

Summary: $u(r, \theta) = R(r)\,\Phi(\theta)$ $\qquad n = 1, 2, 3, \ldots$

$$\Rightarrow \nu = \frac{n^2\pi^2}{\alpha^2}, \quad \Phi_n(\theta) = \sin\frac{n\pi\theta}{\alpha},$$

$$R_n(r) = A_n r^{\frac{n\pi}{\alpha}} + B_n r^{-\frac{n\pi}{\alpha}}$$

$$\boxed{u(r,\theta) = \sum_{n=1}^{\infty}\left(A_n r^{\frac{n\pi}{\alpha}} + B_n r^{-\frac{n\pi}{\alpha}}\right)\sin\left(\frac{n\pi\theta}{\alpha}\right).}$$

F. P. 117

? ? ?

c_n

$r = r_2$

$$f_2(\theta) = \sum_{n=1}^{\infty}\left(A_n r_2^{\frac{n\pi}{\alpha}} + B_n r_2^{-\frac{n\pi}{\alpha}}\right)\sin\frac{n\pi\theta}{\alpha}$$

d_n

$r = r_1$

$$f_1(\theta) = \sum_{n=1}^{\infty}\left(A_n r_1^{\frac{n\pi}{\alpha}} + B_n r_1^{-\frac{n\pi}{\alpha}}\right)\sin\frac{n\pi\theta}{\alpha}$$

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$$f_2(\theta) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi\theta}{\alpha}$$

$$\boxed{C_n = \frac{2}{\alpha}\int_0^{\alpha} f_2(\theta) \sin \frac{n\pi\theta}{\alpha}\, d\theta}$$

$f_1 \ldots -$

$$f_1(\theta) = \sum_{n=1}^{\infty} d_n \sin \frac{n\pi\theta}{\alpha}$$

$$\boxed{d_n = \frac{2}{\alpha}\int_0^{\alpha} f_1(\theta) \sin\left(\frac{n\pi\theta}{\alpha}\right) d\theta}$$

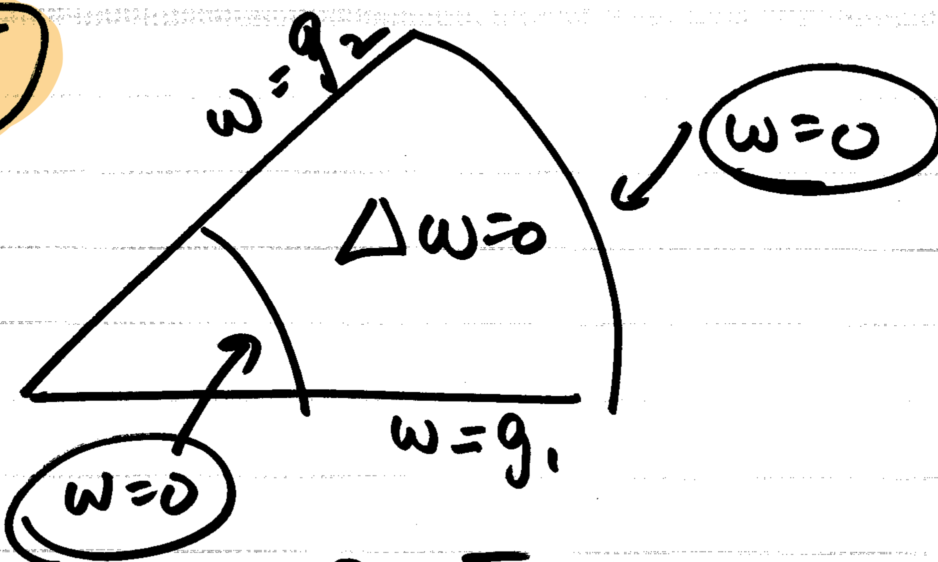
$$\begin{cases} A_n r_2^{\frac{n\pi}{\alpha}} + B_n r_2^{-\frac{n\pi}{\alpha}} = c_n \\ A_n r_1^{\frac{n\pi}{\alpha}} + B_n r_1^{-\frac{n\pi}{\alpha}} = d_n \end{cases}$$

Using Cramer's Rule:

$$A_n = \frac{\begin{vmatrix} c_n & r_2^{-n\pi/\alpha} \\ d_n & r_1^{-n\pi/\alpha} \end{vmatrix}}{\begin{vmatrix} r_2^{n\pi/\alpha} & r_2^{-n\pi/\alpha} \\ r_1^{n\pi/\alpha} & r_1^{-n\pi/\alpha} \end{vmatrix}} = \frac{c_n r_1^{-\frac{n\pi}{\alpha}} - d_n r_2^{-\frac{n\pi}{\alpha}}}{\left(\frac{r_2}{r_1}\right)^{n\pi/\alpha} - \left(\frac{r_2}{r_1}\right)^{-\frac{n\pi}{\alpha}}}$$

$$B_n = \frac{\begin{vmatrix} r_2^{n\pi/\alpha} & c_n \\ r_1^{n\pi/\alpha} & d_n \end{vmatrix}}{\begin{vmatrix} r_2^{n\pi/\alpha} & r_2^{-n\pi/\alpha} \\ r_1^{n\pi/\alpha} & r_1^{-n\pi/\alpha} \end{vmatrix}} = \frac{d_n r_2^{\frac{n\pi}{\alpha}} - c_n r_1^{\frac{n\pi}{\alpha}}}{\left(\frac{r_2}{r_1}\right)^{\frac{n\pi}{\alpha}} - \left(\frac{r_2}{r_1}\right)^{-\frac{n\pi}{\alpha}}}$$

II



Shill $w = R(r)\Phi(\theta)$

Shill
$$\begin{cases} \Phi'' = -\nu\Phi \\ r^2R'' + rR' - \nu R = 0 \end{cases}$$

$R(r_1) = 0$

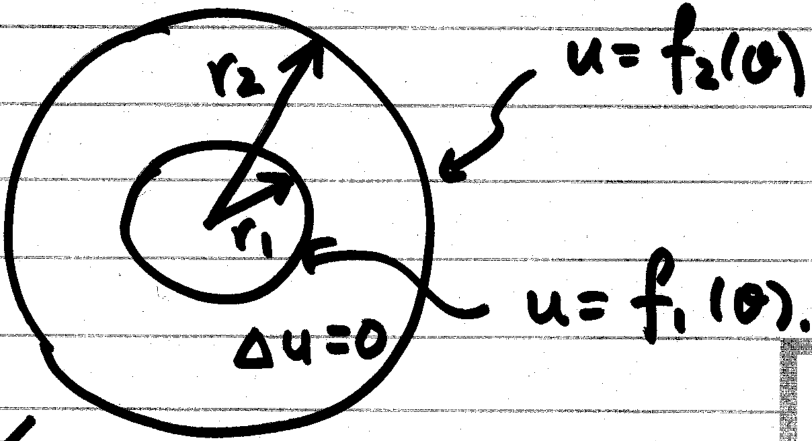
$R(r_2) = 0$

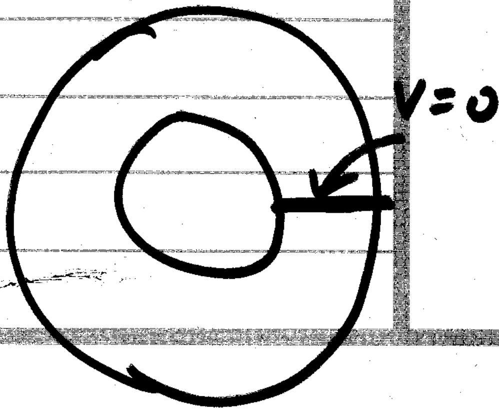
(#7, Pg120; #10 Pg94)

see also additional note.

~~(II) A bit more complicated. (Not)~~

Annulus?





~~$\alpha \to \pi$~~ ~~It is Not the same as~~ $\alpha \to \pi$

Go back to original technique of seperation of variable.

$$\Delta u = \partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_\theta^2 u$$

Try $\quad u(r,\theta) = R(r)\Phi(\theta)$

$$R''\Phi + \frac{1}{r}R'\Phi + \frac{1}{r^2}\Phi'' = 0$$

$\frac{1}{R\Phi}$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Phi''}{\Phi} = 0$$

(r)

$$\frac{r^2R'' + rR'}{R} = -\frac{\Phi''}{\Phi} = \nu$$

(θ)

$$\begin{cases} \Phi'' = -\nu \Phi \leftarrow \text{B.C.?} \\ r^2 R'' + rR' - \nu R = 0 \leftarrow \end{cases}$$

B.C. for Φ: periodic: $\Phi(\theta) = \Phi(2\pi + \theta)$

(Lecture 19) $\begin{cases} \Phi'' = -\nu \Phi \\ \Phi - 2\pi \text{ periodic} \end{cases}$

$\Rightarrow \boxed{\nu = n^2}$ $\quad n = 0, 1, 2, \ldots$

$$\boxed{\begin{array}{l} n = 0, \quad \Phi = 1 \\ n \geq 1, \quad \Phi = \cos n\theta, \sin n\theta. \end{array}}$$

For R: $r^2R'' + rR' - \nu R = 0$ (Euler Equation) (Lecture 29)

$\nu = 0 \Rightarrow r^2R'' + rR' = 0 \Rightarrow rR'' + R' = 0$

$(rR')' = 0$

$rR' = B$

$R' = \dfrac{B}{r}$

$R = A + B\log r$

$$R(r) = A + B\log r$$

$\nu = n^2$ $(n \geq 1)$

$$R(r) = Ar^n + Br^{-n}$$

Summary $\nu = n^2$

$n = 0,$

$\Phi_0(\theta) = 1$

$R_0(r) = A_0 + B_0 \log r$

$n = 1, 2, 3 \ldots$

$\Phi_n(\theta) = \cos n\theta,\ \sin n\theta$

$R_n(r) = Ar^n + Br^{-n}$

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$$u(r,\theta) = R\Phi.$$

F. p. 118, (4.33)

$$\boxed{\begin{aligned} u(r,\theta) &= (A_0 + B_0 \log r)(1) \\ &+ \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n}) \cos n\theta \\ &+ \sum_{n=1}^{\infty} (C_n r^n + D_n r^{-n}) \sin n\theta \end{aligned}}$$

To ~~find coeff~~ !

$r \to r_2 \Rightarrow$

$$f_2(\theta) = A_0 + B_0 \log r_2 + \sum_{n=1}^{\infty} (A_n r_2^n + B_n r_2^{-n}) \cos n\theta + \sum_{n=1}^{\infty} (C_n r_2^n + D_n r_2^{-n}) \sin n\theta$$

const

$r \to r_1 \ldots$

The same as Fourier Series.

at $r=r_2 \Longrightarrow$

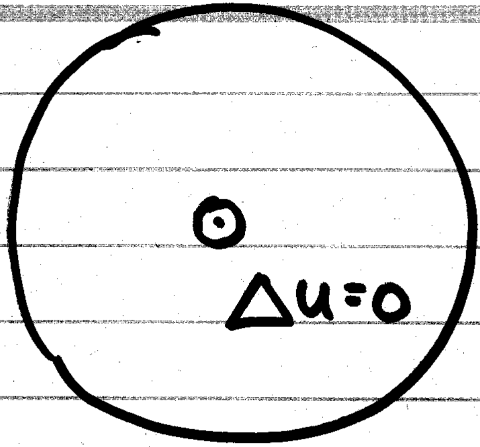
$$\begin{cases} f_2(\theta) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty}\left(a_n \cos n\theta + b_n \sin n\theta\right) \\ \dfrac{a_0}{2} = A_0 + B_0 \log r_2, \\ a_n = A_n r_2^n + B_n r_2^{-n} \\ b_n = C_n r_2^n + D_n r_2^{-n} \end{cases}$$

at $r=r_1 \Longrightarrow$

$$\begin{cases} f_1(\theta) = \dfrac{\tilde{a}_0}{2} + \displaystyle\sum_{n=1}^{\infty}\left(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta\right) \\ \dfrac{\tilde{a}_0}{2} = A_0 + B_0 \log r_1 \\ \tilde{a}_n = A_n r_1^n + B_n r_1^{-n} \\ \tilde{b}_n = C_n r_1^n + D_n r_1^{-n} \end{cases}$$

Solve for $A_0, B_0, A_n, B_n, C_n, D_n$ in terms of a_0, a_n, b_n & $\tilde{a}_0, \tilde{a}_n, \tilde{b}_n$

Circle



$u(r_2,\theta) = f_2(\theta)$.

$r_1 \to 0$

$$\text{Circle}(r_2) = \text{Annulus}(r_2, r_1)$$

Formula for solution in annulus:

$$u(r,\theta) = A_0 + \cancel{B_0 \log r} + \sum_{n=1}^{\infty}\left(A_n r^n + \cancel{B_n r^{-n}}\right)\cos n\theta + \sum_{n=1}^{\infty}\left(C_n r^n + \cancel{D_n r^{-n}}\right)\sin n\theta$$

In order for u to make sense as $r \to 0$

We require $B_0 = 0$, $B_n = 0$, $D_n = 0$

Then,

$$\boxed{u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n r^n \cos n\theta + C_n r^n \sin n\theta}$$

To find A_n's, C_n's, $r \to r_2$

$$f_2(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \underbrace{\left(A_n r_2^n\right)} \cos n\theta + \underbrace{\left(C_n r_2^n\right)} \sin n\theta$$

(Fourier Series)

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$$\left(A_n r_2^n\right) = \frac{1}{\pi}\int_0^{2\pi} f_2(\psi)\cos n\psi\, d\psi$$

$n=0,1,2,\ldots$

$$\left(C_n r_2^n\right) = \frac{1}{\pi}\int_0^{2\pi} f_2(\theta)\sin n\theta\, d\theta$$

$$u(r,\theta) = \underbrace{\frac{1}{2\pi}\int_0^{2\pi}}\underbrace{f_2(\psi)}\,d\psi$$

$$+\sum_{n=1}^{\infty}\left(\frac{r}{r_2}\right)^n\left(\underbrace{\frac{1}{\pi}\int_0^{2\pi}}\underbrace{f_2(\psi)}\cos n\psi\, d\psi\right)\cos n\theta$$

$$+\sum_{n=1}^{\infty}\left(\frac{r}{r_2}\right)^n \underbrace{\frac{1}{\pi}\int_0^{2\pi}}\underbrace{f_2(\psi)}\sin n\psi\, d\psi)\sin n\theta$$

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$$= \int_0^{2\pi} f_2(\psi) \left[\cdots\cdots\cdots \right] d\psi.$$

$$= \frac{1}{2\pi} + \sum_{n=1}^{\infty} \left(\frac{r}{r_2}\right)^n \frac{1}{\pi} \cos n\psi \cos n\theta$$

$$+ \sum_{n=1}^{\infty} \left(\frac{r}{r_2}\right)^n \frac{1}{\pi} \sin n\psi \sin n\theta$$

$$= \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \underbrace{\left(\cos n\psi \cos n\theta + \sin n\psi \sin n\theta \right)}_{\cos[n(\psi-\theta)]} \left(\frac{r}{r_2}\right)^n$$

$$\left(\cos(\alpha-\beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta \right)$$

$u(r,\theta)$ ← Solution

$$= \frac{1}{2\pi}\int_0^{2\pi} f_2(\psi)\left[1 + 2\sum_{n=1}^{\infty}\cos[n(\psi-\theta)]\left(\frac{r}{r_2}\right)^n\right]d\psi$$

$f_2(\psi)$ ↑ B.C.

Poisson Kernel.

$$= \frac{1}{2\pi}\int_0^{2\pi} f_2(\psi)\frac{[r_2^2 - r^2]}{r_2^2 - 2 r r_2 \cos(\theta-\psi) + r^2}\,d\psi.$$

$f_2(\psi)$ ↑ B. value of u at r_2

r_2 ← outer radius

Derivation of Poisson Kernel:

$$\boxed{1 + 2\sum_{n=1}^{\infty}[\cos(n\alpha)]r^n = \frac{1-r^2}{1+r^2-2r\cos\alpha}}$$

$$\boxed{(0 < r < 1)}$$

Pf: $1 + 2\sum_{n=1}^{\infty} r^n \cos(n\alpha)$

$$\boxed{\cos\alpha = \frac{e^{i\alpha}+e^{-i\alpha}}{2}}$$

$$= 1 + 2\sum_{n=1}^{\infty} r^n \left(\frac{e^{in\alpha}+e^{-in\alpha}}{2}\right)$$

$$= 1 + \sum_{n=1}^{\infty} (re^{i\alpha})^n + (re^{-i\alpha})^n$$

- Use geometric series:

$$1+x+x^2+\cdots = \frac{1}{1-x} \quad \text{for } |x|<1$$

$$= 1 + \frac{re^{i\alpha}}{1-re^{i\alpha}} + \frac{re^{-i\alpha}}{1-re^{-i\alpha}}$$

$$= \frac{(1-re^{i\alpha})(1-re^{-i\alpha}) + re^{i\alpha}(1-re^{-i\alpha}) + re^{-i\alpha}(1-re^{i\alpha})}{(1-re^{i\alpha})(1-re^{-i\alpha})}$$

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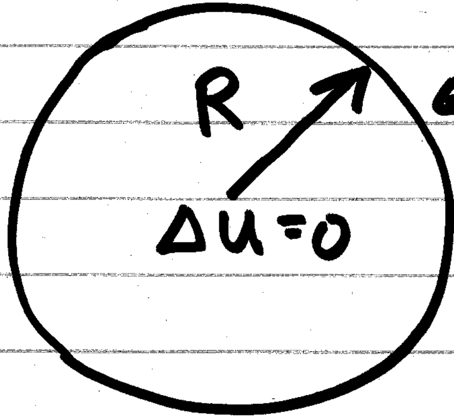
$$= \frac{1 - \cancel{re^{i\alpha}} - \cancel{re^{-i\alpha}} + r^2 + \cancel{re^{i\alpha}} - r^2 + \cancel{re^{-i\alpha}} - r^2}{1 - re^{i\alpha} - re^{-i\alpha} + r^2}$$

$$= \frac{1 - r^2}{1 - r(e^{i\alpha} + e^{-i\alpha}) + r^2}$$

$$= \frac{1 - r^2}{1 - 2r\cos\alpha + r^2} \quad \#$$

Conclusion:

$f(\theta) = u(R,\theta)$



$0 \le r \le R$

$0 \le \theta \le 2\pi$

$$u(r,\theta)\left(= \frac{A_0}{2} + \sum A_n r^n \cos n\theta + \sum C_n r^n \sin n\theta \right).$$

... find A_n's, C_n's

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\psi)\,\frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta-\psi) + r^2}\,d\psi$$

Poisson Kernel

Green's fct for Dir Problem in the circle.