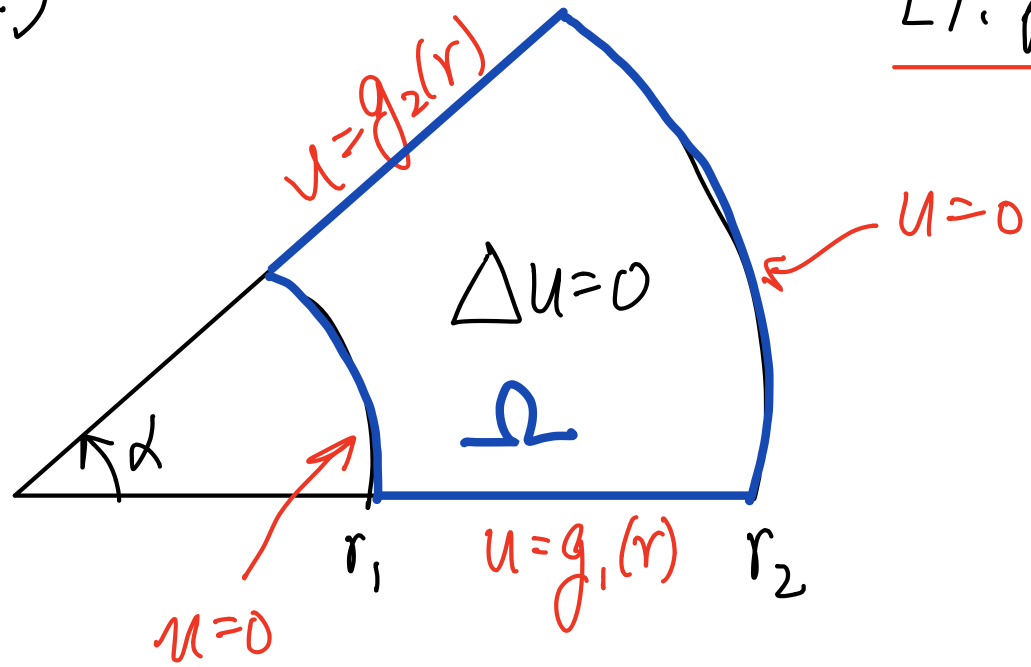
Dirichlet Problem in Circular Sector

(II) [F. p.120, #7]



Ω: (in polar coordinates) $r_1 < r < r_2$, $0 < \theta < \alpha$

$$u = u(r,\theta): \begin{cases} \Delta u = 0 & \text{in } \Omega \\ u(r_1, \theta) = 0, & 0 < \theta < \alpha \\ u(r_2, \theta) = 0, & 0 < \theta < \alpha \\ u(r, 0) = g_1(r), & r_1 < r < r_2 \\ u(r, \alpha) = g_2(r), & r_1 < r < r \end{cases}$$

In polar coordinates:

[F, p.117, (4.29); Appendix 4, p. 404, (A4.3)]

$$\boxed{\begin{aligned} \Delta u &= u_{xx} + u_{yy} \\ &= u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} \end{aligned}}$$

$u = u(x, y)$

$u = u(r, \theta)$

Separation of Variables

$$u(r, \theta) = R(r)\Phi(\theta)$$

$$\Delta u = \left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\right)\left(R(r)\Phi(\theta)\right)$$

$$= R''(r)\Phi(\theta) + \frac{1}{r}R'(r)\Phi(\theta) + \frac{1}{r^2}R(\theta)\Phi''(\theta)$$

$$= 0$$

$$\boxed{\frac{r^2 R''(r) + r R'(r)}{R(r)} = -\frac{\Phi''(\theta)}{\Phi(\theta)} = \nu}$$

constant.

$$\begin{cases} r^2 R''(r) + rR'(r) - \nu R(r) = 0, & \underline{R(r_1) = R(r_2) = 0} \\ \Phi''(\theta) = -\nu \Phi(\theta) & \underline{\nu = ?} \end{cases}$$

$$\boxed{r^2 R''(r) + rR'(r) - \nu R(r) = 0, \quad R(r_1) = R(r_2) = 0}$$

Euler equation. (Can use the theory of Euler Eqn. to solve it.)

[F. p.94 #10] (change of variable)

$$r\left[rR'' + R'\right] - \nu R = 0$$

$$r\partial_r\left[r\partial_r R\right] - \nu R = 0$$

Let $\boxed{s = \log r}$

$$\partial_r = \partial_s \frac{\partial s}{\partial r} = \frac{1}{r}\partial_s$$

Hence $\boxed{\partial_s = r\partial_r}$

Let $R(r)=\tilde{R}(s)$. Then

$$\begin{cases} \partial_s^2 \tilde{R}(s) - \nu \tilde{R}(s) = 0, \\ \tilde{R}(s_1) = \tilde{R}(s_2) = 0, \quad s_1 = \log r_1, \ s_2 = \log r_2 \end{cases}$$

$$\Longrightarrow \boxed{\nu = -\frac{n^2\pi^2}{(s_2-s_1)^2}, \quad \tilde{R}(s) = \sin\left(\frac{n\pi(s-s_1)}{(s_2-s_1)}\right) \quad n = 1, 2, 3, \cdots}$$

Back to $R(r)$:

$$\boxed{\nu = -\frac{n^2\pi^2}{(\log r_2 - \log r_1)^2}, \quad n = 1, 2, 3, \cdots \qquad R(r) = \sin\left(\frac{n\pi(\log r - \log r_1)}{(\log r_2 - \log r_1)}\right)}$$

For $\Phi(\theta)$: $\quad \Phi''(\theta) = -\nu\,\Phi(\theta), \quad 0<\theta<\alpha$

$$\Phi''(\theta) = \frac{n^2\pi^2}{(\log r_2 - \log r_1)^2}\,\Phi(\theta)$$

$$\boxed{\Phi(\theta) = A e^{\frac{n\pi\theta}{(\log r_2 - \log r_1)}} + B e^{-\frac{n\pi\theta}{(\log r_2 - \log r_1)}}}$$

Conclusion

$$\boxed{\begin{aligned} u(r,\theta) &= \sum_{n=1}^{\infty}\left(A_n e^{\frac{n\pi\theta}{(\log r_2 - \log r_1)}} + B_n e^{-\frac{n\pi\theta}{(\log r_2 - \log r_1)}}\right) \times \\ &\qquad \times \sin\left(\frac{n\pi(\log r - \log r_1)}{\log r_2 - \log r_1}\right) \end{aligned}}$$

To find A_n, B_n:

$\theta = 0$:

$$\underline{g_1(r)} = \sum_{n=1}^{\infty} \underbrace{\left(A_n + B_n\right)}_{C_n} \sin\left(\frac{n\pi(\log r - \log r_1)}{\log r_2 - \log r_1}\right)$$

$\theta = \alpha$:

$$\underline{g_2(r)} = \sum_{n=1}^{\infty} \underbrace{\left(A_n e^{\frac{n\pi\alpha}{(\log r_2 - \log r_1)}} + B_n e^{-\frac{n\pi\alpha}{(\log r_2 - \log r_1)}}\right)}_{\tilde{C}_n} \times \sin\left(\frac{n\pi(\log r - \log r_1)}{\log r_2 - \log r_1}\right)$$

Not in the "standard" Fourier Series form.

Use change of variable:

$$\boxed{s = \frac{\log r - \log r_1}{\log r_2 - \log r_1} \qquad 0 < s < 1} \iff \boxed{r = r_1\left(\frac{r_2}{r_1}\right)^s \qquad r_1 < r < r_2}$$

$$g(r) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi(\log r - \log r_1)}{\log r_2 - \log r_1}\right)$$

$$\Updownarrow$$

$\underline{0 \le s \le 1}$

$$g\left(r_1\left(\frac{r_2}{r_1}\right)^s\right) = \sum_{n=1}^{\infty} C_n \sin(n\pi s)$$

$$\Updownarrow$$

$$C_n = 2\int_0^1 g\left(r_1\left(\frac{r_2}{r_1}\right)^s\right)\sin(n\pi s)\,ds$$