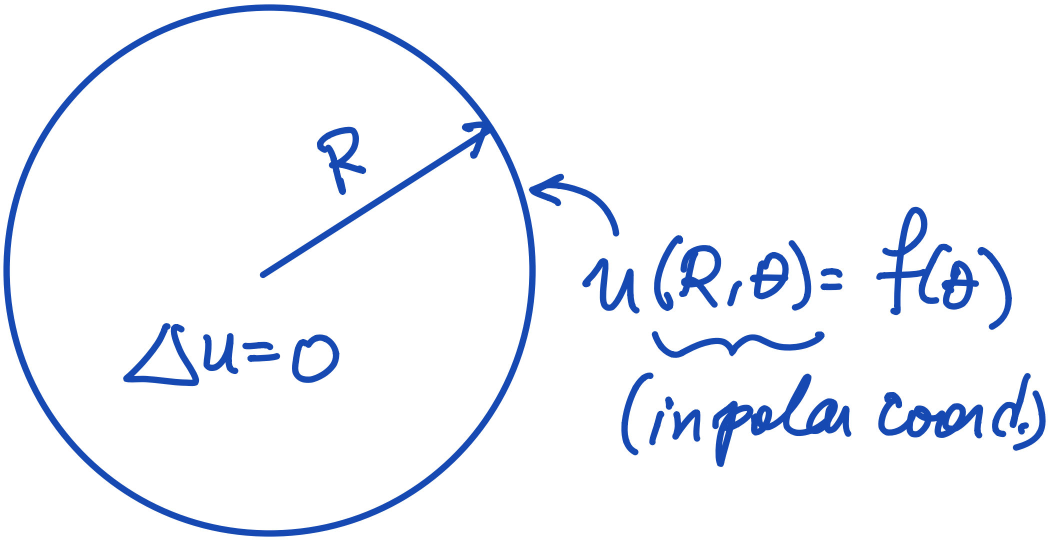
Poisson Kernel for Solving Dirichlet Problem on a Disc [F. p. 118]



$$u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n r^n \cos n\theta + B_n r^n \sin n\theta$$

$$u(R,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n R^n \cos n\theta + B_n R^n \sin n\theta$$

$\Updownarrow$

$$A_0 = \frac{1}{\pi}\int_0^{2\pi} f(\psi)\, d\psi$$

$f(\theta)$:

$$A_n R^n = \frac{1}{\pi}\int_0^{2\pi} f(\theta) \cos n\psi \, d\psi$$

$$B_n R^n = \frac{1}{\pi}\int_0^{\pi} f(\theta) \sin n\psi \, d\psi$$

$$u(r,\theta) = \frac{1}{\pi}\int_0^{2\pi} f(\psi)\frac{1}{2}\,d\psi$$

$$+\sum_{n=1}^{\infty}\left(\frac{1}{\pi}\frac{1}{R^n}\int_0^{2\pi} f(\psi)\cos n\psi\,d\psi\right) r^n \cos n\theta$$

$$+\sum_{n=1}^{\infty}\left(\frac{1}{\pi}\frac{1}{R^n}\int_0^{2\pi} f(\psi)\sin n\psi\,d\psi\right) r^n \sin n\theta$$

$$= \int_0^{2\pi} f(\psi)\left[\cdots\cdots\cdots\cdots\right] d\psi$$

$$\frac{1}{\pi}\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\left(\cos n\psi \cos n\theta + \sin n\psi \sin n\theta\right)\frac{r^n}{R^n}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\left(\cos n(\psi-\theta)\right)\left(\frac{r}{R}\right)^n\right\}$$

$$= \frac{1}{2\pi}\left\{1 + \sum_{n=1}^{\infty} 2\cos(n(\psi-\theta))\left(\frac{r}{R}\right)^n\right\}$$

Define: Poisson Kernel

$$P(r,\theta) = \frac{1}{2\pi}\left[1 + \sum_{n=1}^{\infty} 2\cos(n\theta)\left(\frac{r}{R}\right)^n\right]$$

Then

$$u(r,\theta) = \int_0^{2\pi} f(\psi)\, P(r, \psi-\theta)\, d\psi$$

$$= (f * P)(r,\theta)$$

convolution in θ

Formula for Poisson Kernel

$$P(r,\theta) = \frac{1}{2\pi}\,\frac{R^2 - r^2}{R^2 - 2Rr\cos\theta + r^2} \quad (*)$$

$$u(r,\theta) = \int_0^{2\pi} f(\psi)\,\frac{R^2 - r^2}{R^2 - 2Rr\cos(\psi-\theta) + r^2}\,\frac{d\psi}{2\pi}$$

Pf of (*): $\boxed{1+\sum_{n=1}^{\infty} 2(\cos n\theta)\alpha^n}$ $|\alpha|\leq 1$

$$\left(\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}\right)$$

$$= 1+\sum_{n=1}^{\infty} 2\left(\frac{e^{in\theta}+e^{-in\theta}}{2}\right)\alpha^n$$

$$= 1+\sum_{n=1}^{\infty} \left(e^{in\theta}+e^{-in\theta}\right)\alpha^n$$

$$= 1+\sum_{n=1}^{\infty} (e^{i\theta}\alpha)^n + (e^{-i\theta}\alpha)^n$$

$$\left(1+x+x^2+\cdots = \frac{1}{1-x},\ |x|<1\right)$$

$$= 1+\frac{e^{i\theta}\alpha}{1-e^{i\theta}\alpha}+\frac{e^{-i\theta}\alpha}{1-e^{-i\theta}\alpha}$$

$$= \frac{(1-e^{i\theta}\alpha)(1-e^{-i\theta}\alpha)+e^{i\theta}\alpha(1-e^{-i\theta}\alpha)+e^{-i\theta}\alpha(1-e^{i\theta}\alpha)}{(1-e^{i\theta}\alpha)(1-e^{-i\theta}\alpha)}$$

$$= \frac{1 - e^{i\theta}\alpha - e^{-i\theta}\alpha + \alpha^2 + e^{i\theta}\alpha - \alpha^2 + e^{-i\theta}\alpha - \alpha^2}{1 - e^{i\theta}\alpha - e^{-i\theta}\alpha + \alpha^2}$$

$$= \boxed{\frac{1 - \alpha^2}{1 - 2(\cos\theta)\alpha + \alpha^2}}$$

Set $\alpha = \frac{r}{R} < 1$. Then

$$\boxed{\begin{aligned} P(r,\theta) &= \frac{1}{2\pi} \frac{1 - \left(\frac{r}{R}\right)^2}{1 - 2\cos\theta \frac{r}{R} + \left(\frac{r}{R}\right)^2} \\ &= \frac{1}{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos\theta + r^2} \end{aligned}}$$