How to Solve Heat Equation

1. $$\boxed{\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2},} \qquad x \in (0, L)$$

2. Boundary conditions at $x=0$ & $x=L$

   (i) Dirichlet B.C. : $u(0,t)=0, \quad u(L,t)=0$

   (ii) Neumann B.C. : $u_x(0,t)=0, \quad u_x(L,t)=0$

   (iii) Robin B.C. : $u_x(0,t)=\alpha u(0,t);$ $\quad \alpha, \beta > 0$
   $u_x(L,t) = -\beta u(0,t).$

3. Initial Condition at $t=0$: $u(x,0)=f(x)$

Expansion Using Eigenfunctions

Let $L^2((0,L);\mathbb{R}) = \{ f : \int_0^L |f|^2 dx < \infty \}$

$\mathcal{L} = D\partial_x^2 : \quad \mathcal{L}f = D\partial_x^2 f$

Suppose there is a (complete) base of L^2

$\mathcal{B} = \{ \varphi_1(x), \varphi_2(x), \cdots, \varphi_n(x), \cdots \}$ such that :

(1) $\mathcal{L}\varphi_n(x) = \lambda_n \varphi_n(x)$, i.e. $D\partial_x^2 \varphi_n = \lambda_n \varphi_n$

(2) $\mathcal{B}$ is orthogonal, i.e. $\langle \varphi_n, \varphi_m \rangle = 0, \quad n \neq m$

Expansion Using Eigenfunctions

Write the solution $u = u(x,t)$ as:

$$\boxed{u(x,t) = c_1(t)\varphi_1(x) + c_2(t)\varphi_2(x) + \cdots + c_n(t)\varphi_n(x) + \cdots}$$

$$\boxed{\partial_t u = D\partial_x^2 u}$$

$$\partial_t u = \underline{\dot{c}_1(t)}\varphi_1(x) + \underline{\dot{c}_2(t)}\varphi_2(x) + \cdots + \underline{\dot{c}_n(t)}\varphi_n(x) + \cdots$$

$$D\partial_x^2 u = \underline{\lambda_1 c_1(t)}\varphi_1(x) + \underline{\lambda_2 c_2(t)}\varphi_2(x) + \cdots + \underline{\lambda_n c_n(t)}\varphi_n(x) + \cdots$$

$$\boxed{\dot{c}_n(t) = \lambda_n c_n(t)}$$

Expansion Using Eigenfunctions

Write the solution $u = u(x,t)$ as:

$$\boxed{u(x,t) = c_1(t)\varphi_1(x) + c_2(t)\varphi_2(x) + \cdots + c_n(t)\varphi_n(x) + \cdots}$$

$$\boxed{\dot{c}_n(t) = \lambda_n c_n(t)} \Longrightarrow \boxed{c_n(t) = c_n(0)e^{\lambda_n t}}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} c_n(0)e^{\lambda_n t}\varphi_n(x)}$$

Expansion Using Eigenfunctions

Write the solution $u = u(x,t)$ as:

$$\boxed{u(x,t) = c_1(t)\varphi_1(x) + c_2(t)\varphi_2(x) + \cdots + c_n(t)\varphi_n(x) + \cdots}$$

$$\boxed{\dot{c}_n(t) = \lambda_n c_n(t)} \Longrightarrow \boxed{c_n(t) = c_n(0)e^{\lambda_n t}}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} c_n(0)e^{\lambda_n t}\varphi_n(x)}$$

At $t=0$:

φ_n – orth.

$$\boxed{f(x) = \sum_{n=1}^{\infty} c_n(0)\varphi_n(x)}, \qquad \boxed{c_n(0) = \frac{\langle f, \varphi_n \rangle}{\|\varphi_n\|^2}}$$

Expansion Using Eigenfunctions

Write the solution $u = u(x,t)$ as:

$$\boxed{u(x,t) = c_1(t)\varphi_1(x) + c_2(t)\varphi_2(x) + \cdots + c_n(t)\varphi_n(x) + \cdots}$$

$$\boxed{\dot{c}_n(t) = \lambda_n c_n(t)} \Longrightarrow \boxed{c_n(t) = c_n(0)e^{\lambda_n t}}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} c_n(0)e^{\lambda_n t}\varphi_n(x)}$$

Solution:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{\langle f, \varphi_n \rangle}{\|\varphi_n\|^2} e^{\lambda_n t}\varphi_n(x)}$$

How to find $\varphi_n(x)$ & λ_n's

$$\boxed{D\partial_x^2\varphi(x) = \lambda\varphi(x)}$$

Make use of Boundary Condition !!!

1. $\lambda > 0$: can be eliminated
2. $\lambda = 0$: might happen, depend on B.C.
3. $\lambda < 0$: $\boxed{\varphi(x) = A\cos\left(\sqrt{\frac{-\lambda}{D}}x\right) + B\sin\left(\sqrt{\frac{-\lambda}{D}}x\right)}$

How to find $\varphi_n(x)$ & λ_n's $\boxed{D\partial_x^2\varphi(x)=\lambda\varphi(x)}$

$$\boxed{\varphi(x)=A\cos\left(\sqrt{\frac{-\lambda}{D}}\,x\right)+B\sin\left(\sqrt{\frac{-\lambda}{D}}\,x\right)}\qquad\boxed{\lambda<0}$$

Dirichlet B.C.: $\varphi(0)=0,\ \varphi(L)=0$

$$\boxed{\lambda_n=-\frac{Dn^2\pi^2}{L^2},\quad \varphi_n(x)=\sin\left(\frac{n\pi x}{L}\right),\quad n=1,2,\cdots\cdots}$$

$$u(x,t)=\sum_{n=1}^{\infty}C_n e^{-\frac{Dn^2\pi^2}{L^2}t}\sin\left(\frac{n\pi x}{L}\right),\qquad C_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

How to find $\varphi_n(x)$ & λ_n's

$$\boxed{D\partial_x^2\varphi(x) = \lambda\varphi(x)}$$

$$\boxed{\varphi(x) = A\cos\left(\sqrt{\frac{-\lambda}{D}}x\right) + B\sin\left(\sqrt{\frac{-\lambda}{D}}x\right)}$$

$$\boxed{\lambda \leq 0}$$

Neumann B.C.: $\varphi_x(0) = 0$, $\varphi_x(L) = 0$

$$\boxed{\lambda_n = -\frac{Dn^2\pi^2}{L^2}, \quad \varphi_n(x) = \cos\left(\frac{n\pi x}{L}\right), \quad n = 0, 1, 2 \cdots}$$

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-\frac{Dn^2\pi^2}{L^2}t}\cos\left(\frac{n\pi x}{L}\right), \qquad c_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

How to find $\varphi_n(x)$ & λ_n's

$$\boxed{D\partial_x^2\varphi(x) = \lambda\varphi(x)}$$

$$\boxed{\varphi(x) = A\cos\left(\sqrt{\frac{-\lambda}{D}}x\right) + B\sin\left(\sqrt{\frac{-\lambda}{D}}x\right)}$$

$$\boxed{\lambda < 0}$$

"Mixed" B.C.: $\varphi(0) = 0$, $\varphi_x(L) = 0$

$$\boxed{\lambda_n = -\frac{D(2n-1)^2\pi^2}{4L^2}, \quad \varphi_n(x) = \sin\left(\frac{(2n-1)\pi x}{2L}\right) \quad n = 1, 2 \cdots}$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\frac{D\pi^2(2n-1)^2 t}{4L^2}} \sin\left(\frac{(2n-1)\pi x}{2L}\right)$$

How to find $\varphi_n(x)$ & λ_n's

$$\boxed{D\,\partial_x^2 \varphi(x) = \lambda \varphi(x)}$$

$$\boxed{\varphi(x) = A\cos\left(\sqrt{\frac{-\lambda}{D}}\,x\right) + B\sin\left(\sqrt{\frac{-\lambda}{D}}\,x\right)} \qquad \boxed{\lambda < 0}$$

"Mixed" B.C.: $\varphi(0) = 0, \quad \varphi_x(L) = 0$

$$\boxed{\lambda_n = -\frac{D(2n-1)^2\pi^2}{4L^2}, \quad \varphi_n(x) = \sin\left(\frac{(2n-1)\pi x}{2L}\right) \quad n = 1, 2 \cdots}$$

$$C_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{(2n-1)\pi x}{2L}\right)dx$$

How to find $\varphi_n(x)$ & λ_n's $\boxed{D\partial_x^2\varphi(x) = \lambda\varphi(x)}$

$$\boxed{\varphi(x) = A\cos\left(\sqrt{\frac{-\lambda}{D}}x\right) + B\sin\left(\sqrt{\frac{-\lambda}{D}}x\right)} \qquad \boxed{\lambda < 0}$$

Robin's B.C.: $\varphi_x(0) = \alpha\varphi(0)$; $\varphi_x(L) = -\beta\varphi(L)$

$(\alpha, \beta > 0)$

Let $\nu = \sqrt{\frac{-\lambda}{D}}$. Then ν must solve:

$$\boxed{\tan(\nu L) = \frac{\nu(\alpha+\beta)}{\nu^2 - \alpha\beta}} \qquad [F, p. 91] \quad (3.37)$$

$$\boxed{\varphi(x) = A\cos\left(\sqrt{\frac{-\lambda}{D}}x\right) + B\sin\left(\sqrt{\frac{-\lambda}{D}}x\right)}$$

$\boxed{\lambda < 0}$

Robin's B.C.: $\varphi_x(0) = \alpha\varphi(0)$; $\varphi_x(L) = -\beta\varphi(L)$

$(\alpha, \beta > 0)$

$$\varphi(x) = A\cos(\nu x) + B\sin(\nu x)$$

$$\varphi_x(x) = -A\nu\sin(\nu x) + B\nu\cos(\nu x)$$

$x = 0 \Rightarrow \quad B\nu = \alpha A$

$x = L \Rightarrow \quad -A\nu\sin(\nu L) + B\nu\cos(\nu L)$

$$= -\beta[A\cos(\nu L) + B\sin(\nu L)]$$

$$\boxed{\varphi(x) = A\cos\left(\sqrt{\frac{-\lambda}{D}}\,x\right) + B\sin\left(\sqrt{\frac{-\lambda}{D}}\,x\right)} \qquad \boxed{\lambda < 0}$$

Robin's B.C.: $\varphi_x(0) = \alpha\varphi(0)$; $\varphi_x(L) = -\beta\varphi(L)$

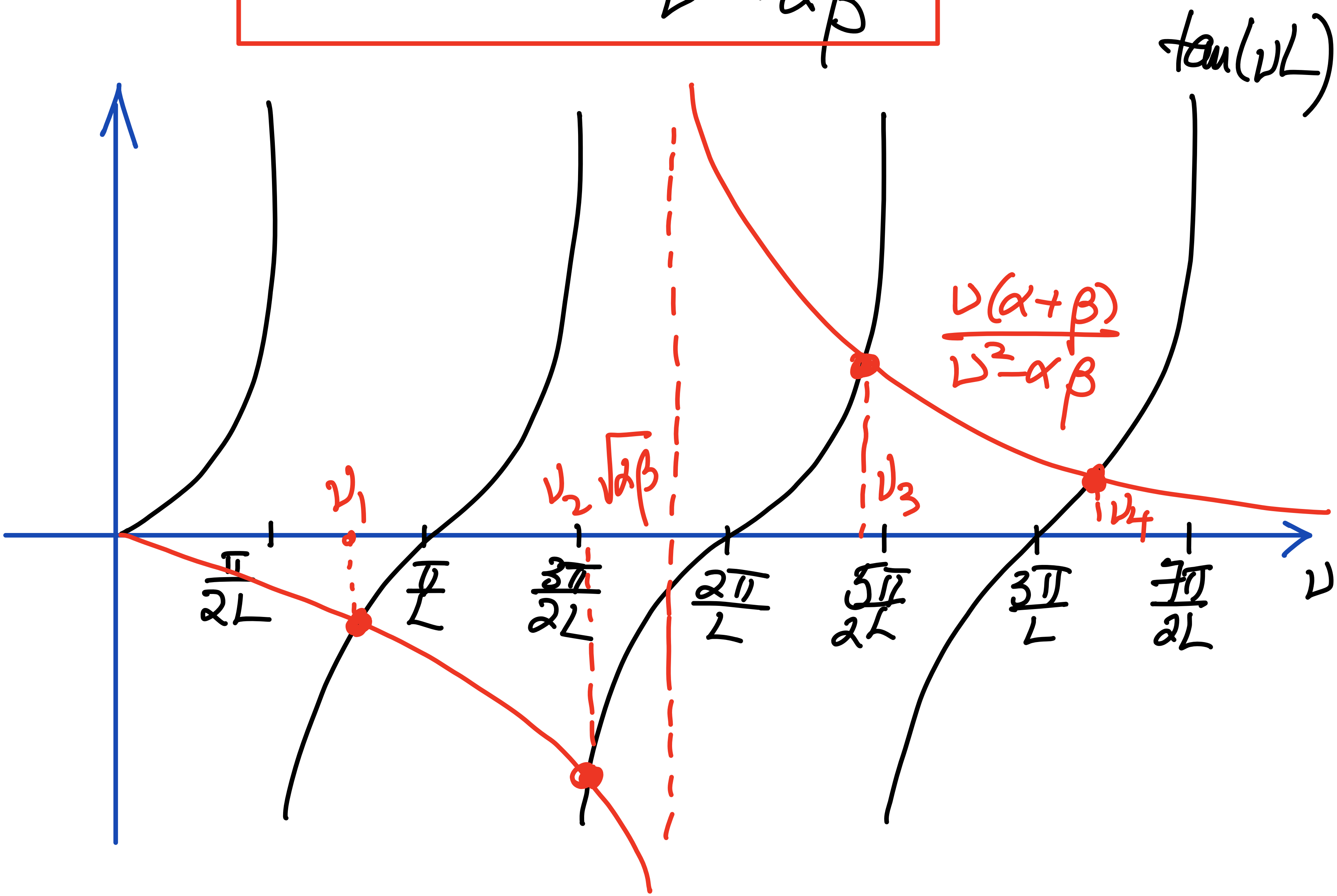
$(\alpha, \beta > 0)$

$$-\frac{\cancel{B}\nu^2}{\alpha}\sin(\nu L) + \cancel{B}\nu\cos(\nu L) = -\beta\left[\frac{\cancel{B}\nu}{\alpha}\cos(\nu L) + \cancel{B}\sin(\nu L)\right]$$

$$\Rightarrow \left(-\frac{\nu^2}{\alpha} + \beta\right)\sin(\nu L) = -\nu\left(\frac{\beta}{\alpha} + 1\right)\cos(\nu L)$$

$$\Rightarrow \quad \boxed{\tan(\nu L) = \frac{\nu(\alpha+\beta)}{\nu^2 - \alpha\beta}} \qquad \begin{array}{c}\nu > 0 \\ (\lambda < 0)\end{array}$$

$$\boxed{\tan(\nu L) = \frac{\nu(\alpha+\beta)}{\nu^2-\alpha\beta}}$$



How to find $\varphi_n(x)$ & λ_n's

$$\boxed{D\partial_x^2\varphi(x) = \lambda\varphi(x)}$$

$$\boxed{\varphi(x) = A\cos\left(\sqrt{\frac{-\lambda}{D}}\,x\right) + B\sin\left(\sqrt{\frac{-\lambda}{D}}\,x\right)}$$

$$\boxed{\lambda < 0}$$

Robin's B.C.: $\varphi_x(0) = \alpha\varphi(0)$; $\varphi_x(L) = -\beta\varphi(L)$

$(\alpha, \beta > 0)$

There are infinitely many solutions for ν:

$$\{\nu_1, \nu_2, \nu_3, \ldots\ldots\} \quad (\nu_i > 0)$$

$$\boxed{\lambda_n = -D\nu_n^2, \qquad \varphi_n(x) = \nu_n\cos(\nu_n x) + \alpha\sin(\nu_n x)}$$

How to find $\varphi_n(x)$ & λ_n's

$$\boxed{D\partial_x^2\varphi(x) = \lambda\varphi(x)}$$

$$\boxed{\varphi(x) = A\cos\left(\sqrt{\frac{-\lambda}{D}}x\right) + B\sin\left(\sqrt{\frac{-\lambda}{D}}x\right)} \qquad \boxed{\lambda < 0}$$

Robin's B.C.: $\varphi_x(0) = \alpha\varphi(0)$; $\varphi_x(L) = -\beta\varphi(L)$

$(\alpha, \beta > 0)$

$$u_t = D u_{xx}, \qquad x \in (0, L)$$

$$u(x, 0) = f(x)$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-D\nu_n^2 t}\underbrace{\left(\nu_n\cos(\nu_n x) + \alpha\sin(\nu_n x)\right)}_{\varphi_n(x)}$$

How to find $\varphi_n(x)$ & λ_n's

$$\boxed{D\partial_x^2\varphi(x) = \lambda\varphi(x)}$$

$$\boxed{\varphi(x) = A\cos\left(\sqrt{\frac{-\lambda}{D}}x\right) + B\sin\left(\sqrt{\frac{-\lambda}{D}}x\right)} \qquad \boxed{\lambda < 0}$$

Robin's B.C.: $\varphi_x(0) = \alpha\varphi(0)$; $\varphi_x(L) = -\beta\varphi(L)$

$(\alpha, \beta > 0)$

$$C_n = \frac{\langle f, \varphi_n\rangle}{\|\varphi_n\|^2} = \frac{\int_0^L f(x)\left(\nu_n\cos\nu x + \alpha\sin(\nu x)\right)dx}{\frac{1}{2}(\nu^2+\alpha^2)L + \left(\frac{\nu^2-\alpha^2}{2\nu}\right)\cos\nu L\sin\nu L + \alpha\sin^2(\alpha L)}$$

[F, p.91]

An Important Identity

Given any 2 functions $f(x)$ & $g(x)$,

$$\int_0^L f_{xx}(x)\, g(x)\, dx = -\int_0^L f_x(x)\, g_x(x)\, dx \;(+B.C.)$$

$$= \int_0^L f(x)\, g_{xx}(x)\, dx \;(+B.C.)$$

For any function $f(x)$,

$$\int_0^L f_{xx}(x)\, f(x) = -\int_0^L f_x^2(x)\, dx \;(+B.C.)$$

An Important Identity

Consequences: (with suitable B.C.)

① $\lambda \leq 0$

related to dissipation mechanism of $\mathcal{L} = D\partial_x^2$

② $\varphi_n \perp \varphi_m$ if $\lambda_n \neq \lambda_m$

related to eigenvectors of symmetric operator are orthogonal to each other

Completeness of Trigonometric Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

(2π-periodic)

(1) Pointwise / Uniform convergence:

$$\lim_{N\to\infty} \left| f(x) - \left(\frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos nx + b_n \sin nx \right) \right| = 0$$

f is required to have "derivative".

(Proof using Dirichlet Kernel.)

Completeness of Trigonometric Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

(2π-periodic)

(2) L^2- Convergence (Convergence in L^2-norm)

Function space:

$$\boxed{L^2((-\pi,\pi):\mathbb{R}) = \left\{ f : \int_{-\pi}^{\pi} |f(x)|^2 dx < \infty \right\}}$$

Completeness of Trigonometric Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

(2π-periodic)

(2) L^2- Convergence (Convergence in L^2-norm)

$\boxed{\mathcal{B} = \{1, \cos nx, \sin nx, n = 1, 2, \cdots\} \text{ is complete (orthogonal) basis in } L^2((-\pi,\pi), \mathbb{R})}$

(Proof using projection, Bessel's Inequality & approximation)

Completeness of Trigonometric Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

(2π-periodic)

(2) L^2- Convergence (Convergence in L^2-norm)

$$S_N f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$$

$$\| f - S_N f \|_{L^2}^2 = \int_{-\pi}^{\pi} |f(x) - S_N f(x)|^2 dx \xrightarrow{N \to +\infty} 0$$

Completeness of Eigenfunctions

(See [F, Thm 3.9, Thm 3.10] p. 89–90 for more detail)

$$\boxed{\mathcal{L}u = (r(x) f_x(x))_x + p(x) f(x)} \quad (+\text{B.C.}) \qquad x \in (0, L)$$

Thm 3.9 : (a) All eigenvalues of $\mathcal{L}$ are real numbers;

(b) Eigenfunctions corresponding to distinct eigenvalues are orthogonal.

Thm 3.10 : (a) $\lambda_n \longrightarrow -\infty$

(b) Eigenfunctions form a complete orthogonal basis of $L^2((0, L), \mathbb{R})$