Solution of Heat Equation

$$\boxed{\begin{aligned} &\partial_t u = D u_{xx}, \quad x \in (0,L), \\ &u(x,0) = f(x), \quad + \text{homog. B.C.} \end{aligned}}$$

$\leftarrow$ homog. eqn.

(Let $\{(\varphi_n, \lambda_n)\}_{n\geq 1}$ be pairs of eigenfunctions & eigenvalues. The $\{\varphi_n\}_{n\geq 1}$ forms a (complete orthogonal) basis of L^2.)

$$\boxed{u(x,t) = \sum_n c_n(t)\,\varphi_n(x) = \sum_n c_n(0)\,e^{\lambda_n t}\varphi_n(x)}$$

$$c_n(0) = \frac{\langle f, \varphi_n \rangle}{\|\varphi_n\|^2}$$

Solution of Inhomogeneous Heat Equation

$$\boxed{\begin{aligned} &\partial_t u = D u_{xx} + \underline{F(x,t)}, \qquad x \in (0, L) \\ &u(x,0) = f(x), \qquad + \text{homog. B.C.} \end{aligned}}$$

<u>Write</u>: $u(x,t) = \sum_n c_n(t)\,\varphi_n(x)$ $\leftarrow$ $c_n(t) = ?$

$$F(x,t) = \sum_n b_n(t)\,\varphi_n(x)$$ $\leftarrow$ given

$\Longrightarrow$

$$\underbrace{\sum_n \dot{c}_n(t)\,\varphi_n(x)}_{\partial_t u} = \underbrace{\sum_n \lambda_n c_n(t)\,\varphi_n(x)}_{D\partial_{xx} u} + \underbrace{\sum_n b_n(t)\,\varphi_n(x)}_{F(x,t)}$$

Solution of Inhomogeneous Heat Equation

$$\boxed{\dot{c}_n(t) = \lambda_n c_n(t) + b_n(t)}$$

$$\dot{c}_n(t) - \lambda_n c_n(t) = b_n(t)$$

$$e^{-\lambda_n t}\dot{c}_n(t) - \lambda_n e^{-\lambda_n t} c_n(t) = e^{-\lambda_n t} b_n(t)$$

$$\frac{d}{dt}\left(e^{-\lambda_n t} c_n(t)\right) = e^{-\lambda_n t} b_n(t)$$

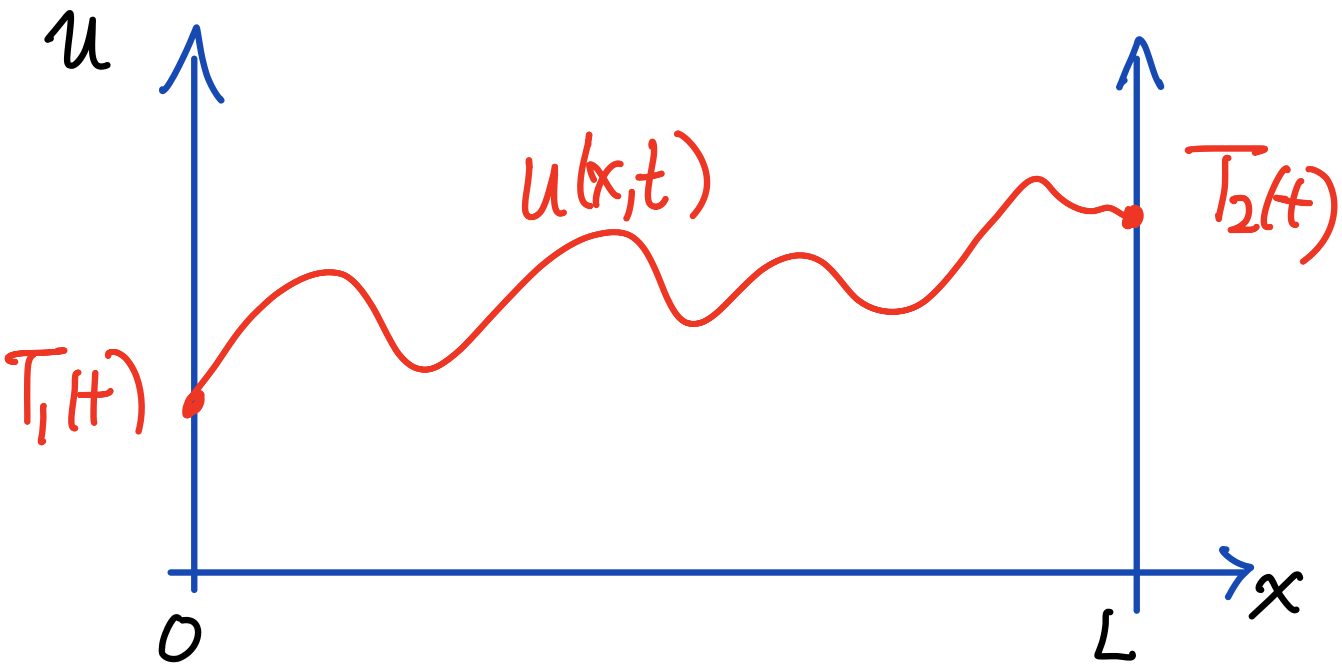
$$e^{-\lambda_n t} c_n(t) - c_n(0) = \int_0^t e^{-\lambda_n s} b_n(s)\, ds$$

$$\boxed{c_n(t) = e^{\lambda_n t} c_n(0) + \int_0^t e^{\lambda_n (t-s)} b_n(s)\, ds}$$

Solution of Inhomogeneous Heat Equation

$$\boxed{\begin{array}{l} \partial_t u = D u_{xx} + \underline{F(x,t)}, \qquad x \in (0, L) \\ u(x,0) = f(x), \quad + \text{inhomog. B.C.} \end{array}}$$

Example: $\underline{u(0,t) = T_1(t)}$, $\underline{u(L,t) = T_2(t)}$

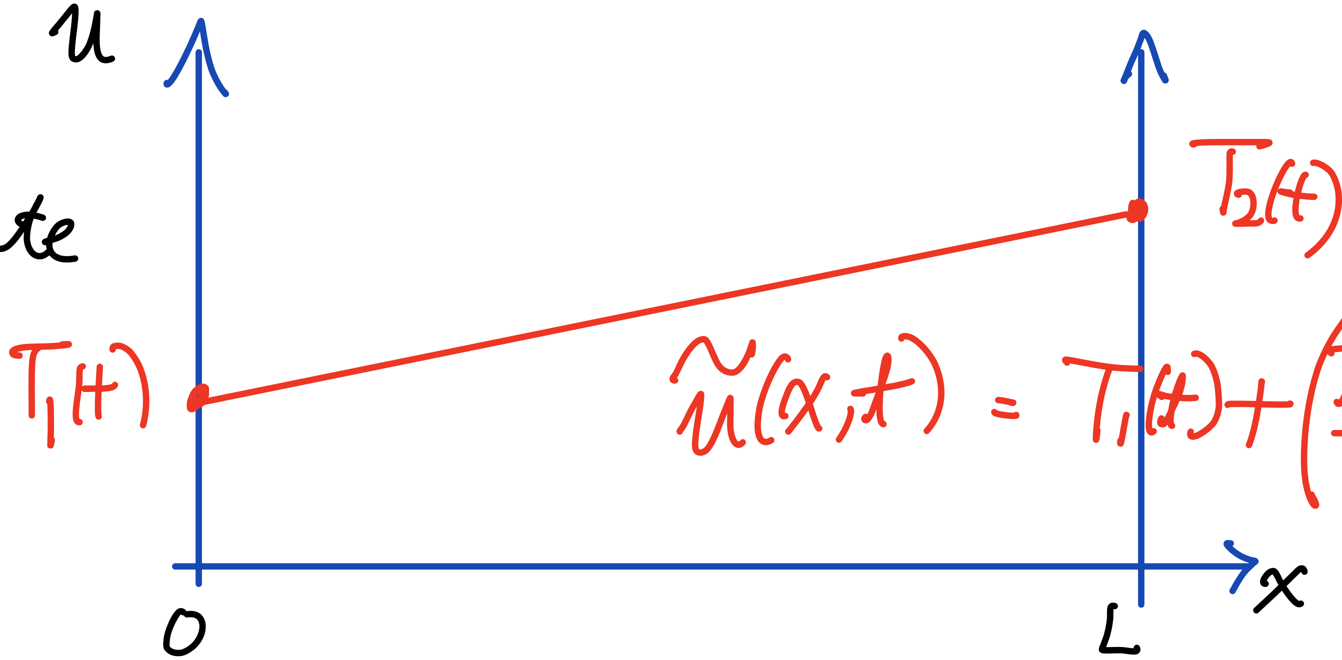


Solution of Inhomogeneous Heat Equation

$$\boxed{\begin{aligned} &\partial_t u = D u_{xx} + \underline{F(x,t)}, \qquad x \in (0, L) \\ &u(x,0) = f(x), \quad + \text{inhomog. B.C.} \end{aligned}}$$

Example: $\underline{u(0,t) = T_1(t)}$, $\underline{u(L,t) = T_2(t)}$

find the steady state $\tilde{u}$



$$\tilde{u}(x,t) = T_1(t) + \left(\frac{T_2(t) - T_1(t)}{L}\right) x$$

Solution of Inhomogeneous Heat Equation

$$\boxed{\begin{aligned} &\partial_t u = D u_{xx} + \underline{F(x,t)}, \qquad x \in (0, L) \\ &u(x,0) = f(x), \quad + \text{inhomog. B.C.} \end{aligned}}$$

Example: $\underline{u(0,t) = T_1(t)}$, $\underline{u(L,t) = T_2(t)}$

Let $u(x,t) = v(x,t) + \tilde{u}(x,t)$

Then (1) $v(0,t) = 0$ & $v(L,t) = 0 \leftarrow$ homog. B.C.

(2) $v(x,0) = u(x,0) - \tilde{u}(x,0)$

$$= f(x) - \tilde{u}(x,0)$$

Solution of Inhomogeneous Heat Equation

$$\boxed{\begin{aligned} &\partial_t u = D u_{xx} + \underline{F(x,t)}, \qquad x \in (0, L) \\ &u(x,0) = f(x), \quad + \text{inhomog. B.C.} \end{aligned}}$$

Example: $\underline{u(0,t) = T_1(t)}$, $\underline{u(L,t) = T_2(t)}$

Let $u(x,t) = v(x,t) + \tilde{u}(x,t)$

Then (3) $\partial_t u = D u_{xx} + F(x,t)$

$$\partial_t v + \partial_t \tilde{u} \qquad D v_{xx} + \cancel{D \tilde{u}_{xx}} + F(x,t)$$

0 (because $\tilde{u}$ is a straight line)

Solution of Inhomogeneous Heat Equation

$$\boxed{\begin{aligned} &\partial_t u = D u_{xx} + \underline{F(x,t)}, \qquad x \in (0, L) \\ &u(x,0) = f(x), \quad + \text{inhomog. B.C.} \end{aligned}}$$

Example: $\underline{u(0,t) = T_1(t)}$, $\underline{u(L,t) = T_2(t)}$

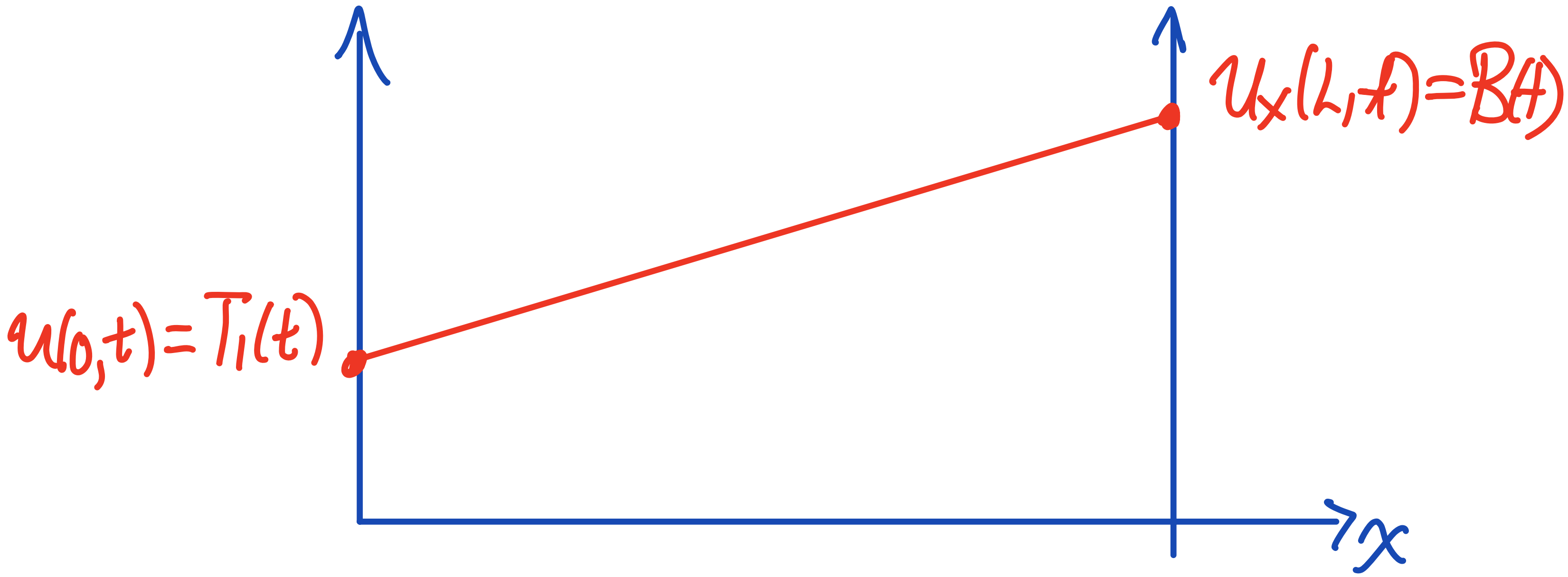
Let $u(x,t) = v(x,t) + \tilde{u}(x,t)$

Then (3) v solves:

$$\boxed{\begin{aligned} &\partial_t v = D v_{xx} + F(x,t) - \partial_t \tilde{u}(x,t) \\ &v(x,0) = f(x) - \tilde{u}(x,0), \quad v(0,t) = v(L,t) = 0 \end{aligned}}$$

Other Inhomogeneous Boundary Conditions

(1) Mixed: $u(0,t) = T_1(t)$, $u_x(L,t) = B(t)$

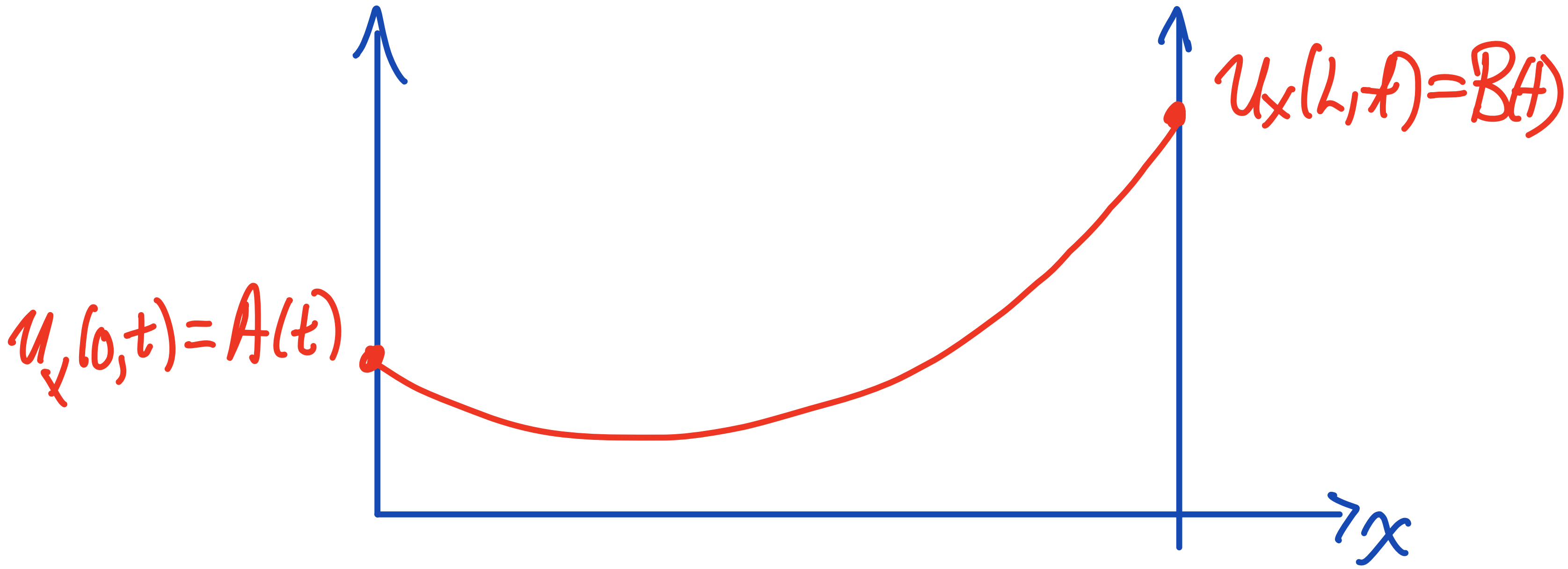


$$\boxed{\tilde{u}(x,t) = T_1(t) + B(t)x}$$

Then $\left(\tilde{u}(x,t)\right)_{xx} = 0$, $\left(\tilde{u}(x,t)\right)_t = \dot{T}_1(t) + \dot{B}(t)x$

Other Inhomogeneous Boundary Conditions

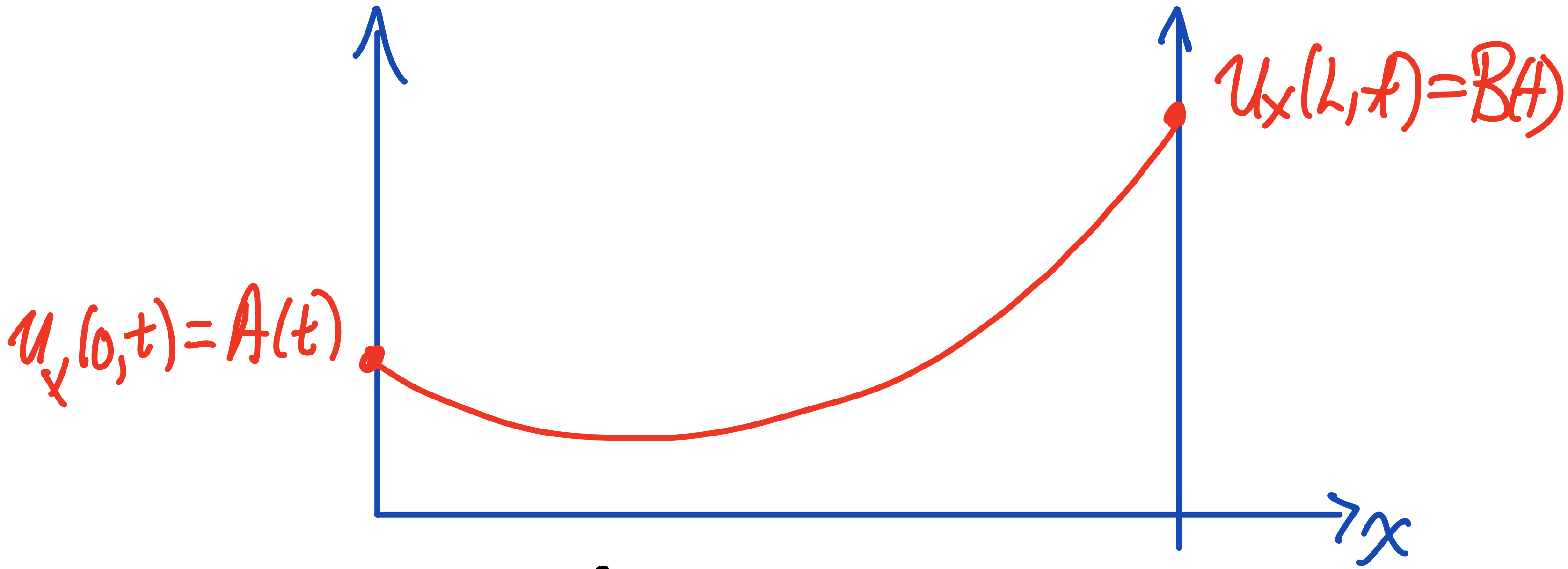
(2) Neumann: $u_x(0,t) = A(t)$, $u_x(L,t) = B(t)$



$$\boxed{\tilde{u}(x,t) = \left(\frac{B(t)-A(t)}{2L}\right)x^2 + A(t)\,x}$$

Other Inhomogeneous Boundary Conditions

(2) Neumann: $u_x(0,t) = A(t)$, $u_x(L,t) = B(t)$



then

$$\tilde{u}_t(x,t) = \left(\frac{\dot{B}(t) - \dot{A}(t)}{2L}\right)x^2 + \dot{A}(t)x$$

$$\tilde{u}_{xx}(x,t) = \left(\frac{B(t) - A(t)}{L}\right)$$