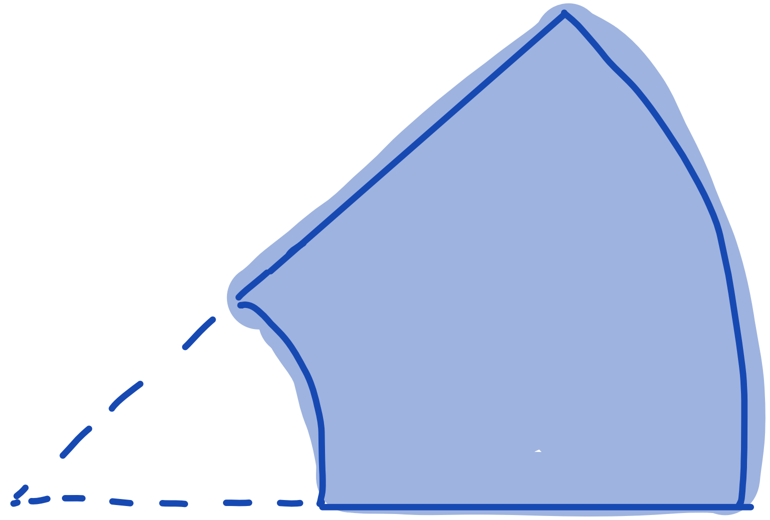
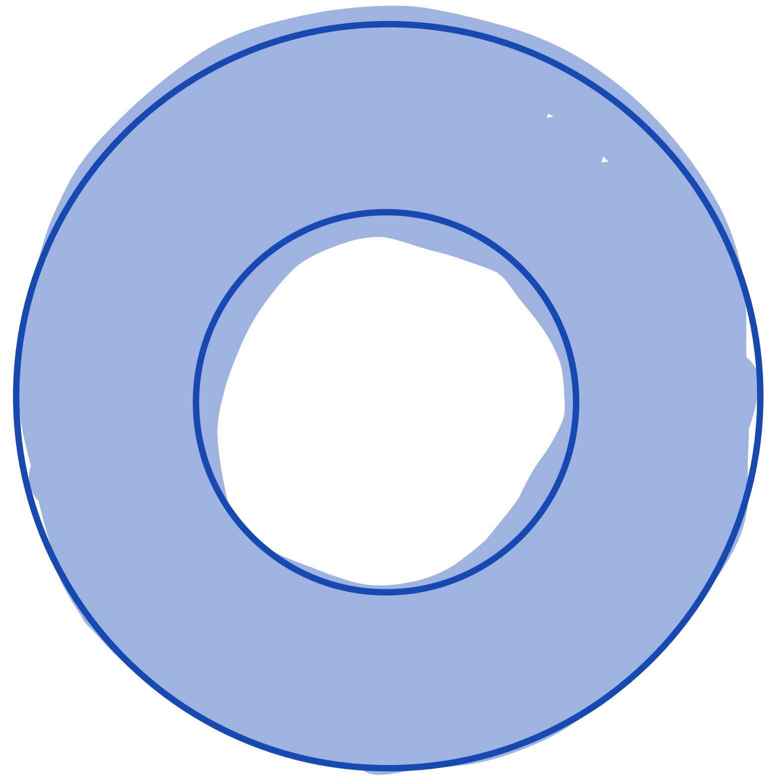
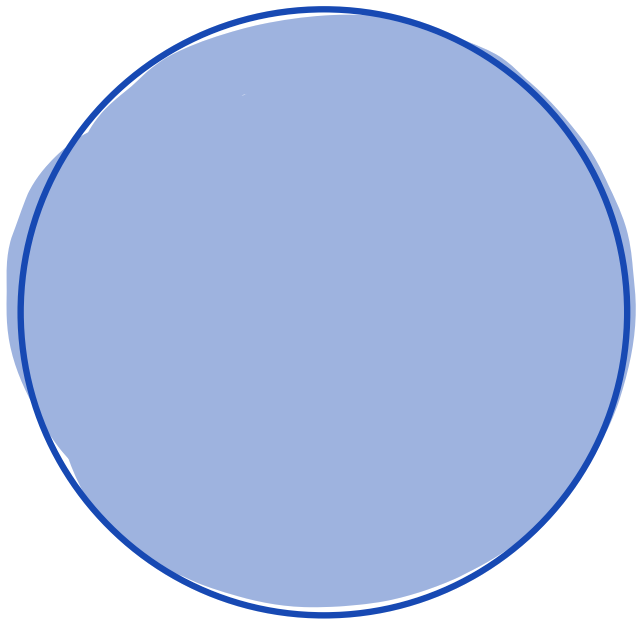
# Eigenfunctions and Eigenvalues of $\Delta$ in Circular Domains







$$\Omega: \begin{array}{l} r_1 < r < r_2 \\ \alpha < \theta < \beta \end{array}$$

# Eigenfunctions and Eigenvalues of $\Delta$ in Circular Domains

$$\Delta u = \lambda u \qquad \lambda = -\mu^2 < 0$$

In $(x,y)$:

$$\Delta u = (\partial_x^2 + \partial_y^2) u$$

In $(r,\theta)$:

$$\Delta u = \left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\right) u$$

$$u = u(r,\theta) = R(r)\Phi(\theta):$$

① $\Phi''(\theta) + \nu^2 \Phi(\theta) = 0$

② $r^2 R''(r) + r R'(r) + (\mu^2 r^2 - \nu^2) R(r) = 0$

# Eigenfunctions and Eigenvalues of $\Delta$ in Circular Domains

$$\Delta u = \lambda u \qquad \lambda = -\mu^2 \leq 0$$

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In $(r,\theta)$:

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① $\Phi''(\theta) + \nu^2 \Phi(\theta) = 0$

$\nu = ?$

$\boxed{B.C.}$ !!!

$\nu = 0$: $\Phi(\theta) = A$;

$\nu > 0$: $\Phi(\theta) = A\cos(\nu\theta) + B\sin(\nu\theta)$

# Eigenfunctions and Eigenvalues of $\Delta$ in Circular Domains

$$\Delta u = \lambda u \qquad \lambda = -\mu^2 \leq 0$$

In $(x, y)$:

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In $(r, \theta)$:

$$\Delta u = \left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\right) u$$

(2) $$\boxed{r^2 R''(r) + r R'(r) + (\mu^2 r^2 - \nu^2) R(r) = 0}$$

$x = \mu r$, $R(r) = f(x) = f(\mu r)$ $\quad \mu = ?$

$$\boxed{x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0}$$

Bessel Eqn.

# Solution of Bessel Equation : Bessel Function

$$\boxed{x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0}$$

$$f(x) = A \underline{J_\nu(x)} + B \underline{Y_\nu(x)}$$

$J_\nu(x) =$ Bessel function of $1^{st}$ Kind

$$Y_\nu(x) \begin{cases} \nu^2 \neq n^2: & \dfrac{(\cos \nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)} \\ \nu = n: & \lim\limits_{\nu \to n} Y_\nu(x) \end{cases}$$

Bessel fct of $2^{nd}$ Kind

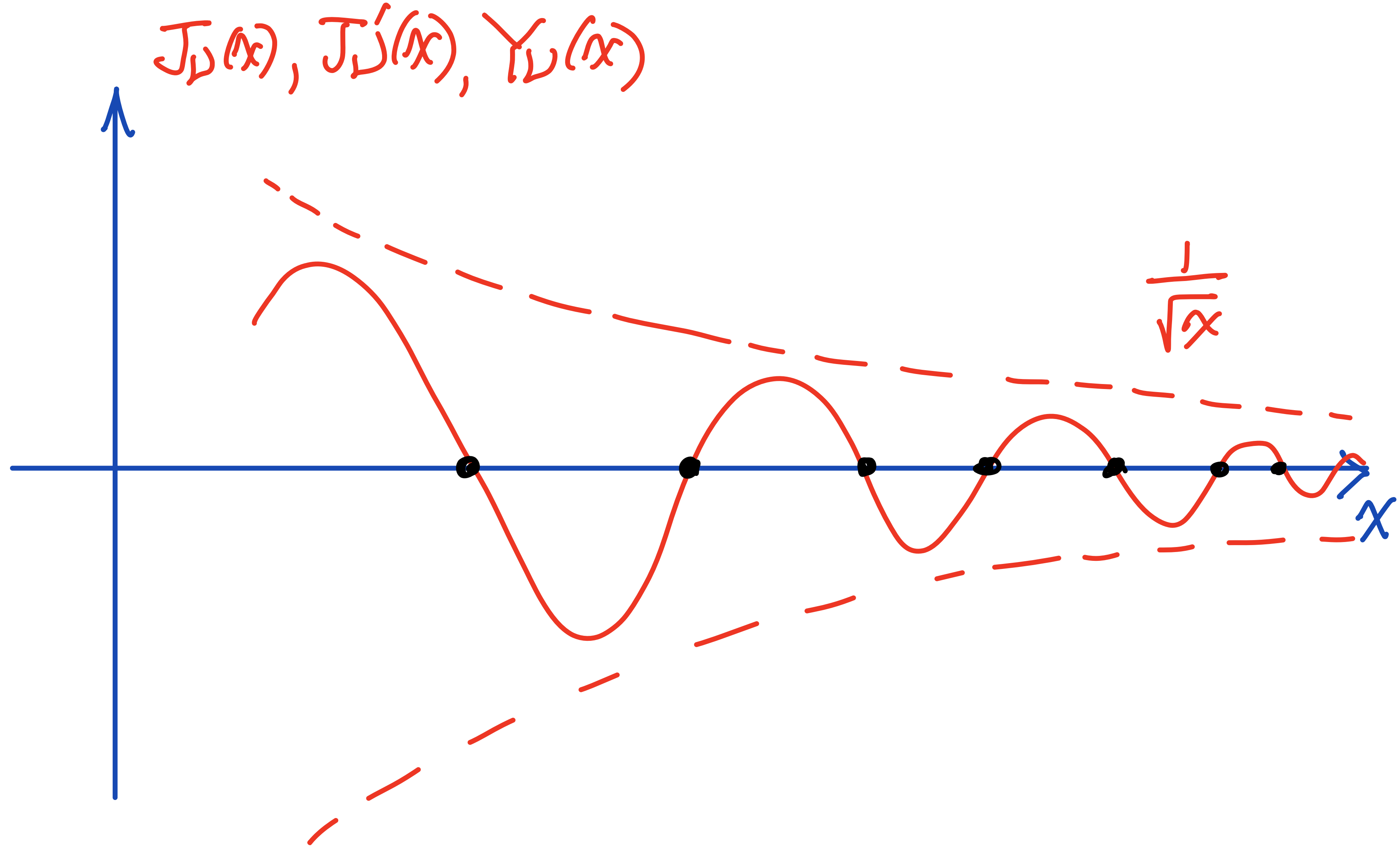
# Asymptotic Shape of Bessel Functions

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{x^{3/2}}\right) \quad (5.25)$$

$$Y_\nu(x) = \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{x^{3/2}}\right) \quad (5.26)$$

$$J_\nu'(x) = -\sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{x^{3/2}}\right) \quad (5.27)$$

# Asymptotic Shape of Bessel Functions



# Zeros of Bessel Functions

Thm 5.2 p.142 $\quad a, b \geq 0,\ (a,b) \neq (0,0)$

Consider $\boxed{a J_\nu(x) + b x J_\nu'(x) = 0}$ $(*)$

Let $\lambda_1, \lambda_2, \lambda_3 \cdots$ be positive zeros of $(*)$

(a) $\lambda_1 > \nu$

(b) If $b = 0$, then there is an integer $M = M(\nu)$,

$$\boxed{\lambda_k \sim \left(k + M + \frac{\nu}{2} + \frac{3}{4}\right)\pi \quad \text{as } k \to +\infty}$$

# Zeros of Bessel Functions

Thm 5.2 p.142 $\quad a, b \geq 0, \ (a,b) \neq (0,0)$

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Let $\lambda_1, \lambda_2, \lambda_3 \cdots$ be positive zeros of $(*)$

(a) $\lambda_1 > \nu$

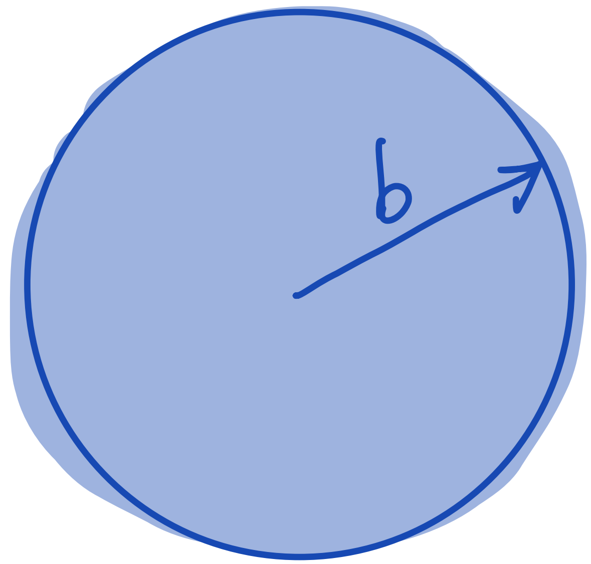
(c) If $b > 0$, then there is an integer $M(\nu, \frac{a}{b})$

$$\boxed{\lambda_k \sim \left(k + M + \frac{\nu}{2} + \frac{1}{4}\right)\pi \quad \text{as } k \to +\infty}$$

$$\Delta u = \lambda u \quad \text{in } D_b = \{x^2+y^2<b^2\}$$

$0<r<b$

$0<\theta<2\pi$




① $\Phi''(\theta)+\nu^2\Phi(\theta)=0$

$\Phi(\theta+2\pi)=\Phi(\theta)$

$\nu^2=n^2$, $\Phi(\theta)=1,\ \cos n\theta,\ \sin n\theta$

② $x^2f''(x)+xf'(x)+(x^2-n^2)f(x)=0$

$f(x)=AJ_n(x)+BY_n(x)$

$R(0^+)$ exists and is finite.

$R(r)=f(\mu r)=AJ_n(\mu r)+B\cancel{Y_n(\mu r)}$

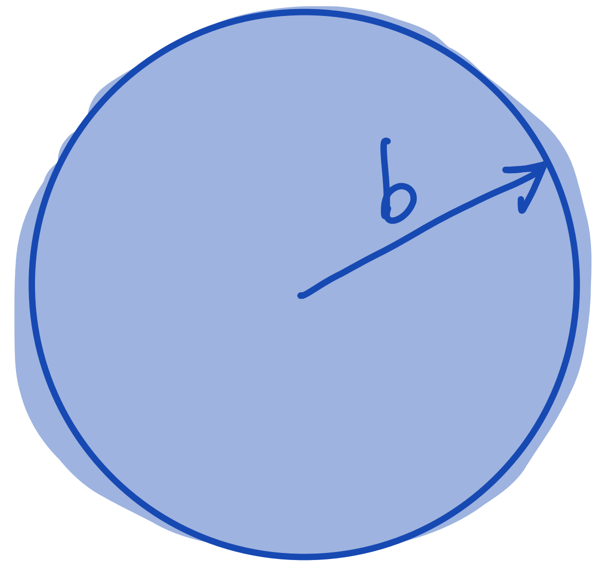
$(\lambda=-\mu^2)$

$\left(B=0 \text{ as } Y_n(\mu r)\xrightarrow[r\to 0]{}-\infty\right)$

$$\Delta u = \lambda u \quad \text{in } D_b = \{x^2+y^2 < b^2\}$$

$0 < r < b$

$0 < \theta < 2\pi$




② $R(r) = A J_n(\mu r)$

$(\lambda = -\mu^2)$

Boundary condition at $r=b$:

Dirichlet B.C. $\quad R(b) = 0$

$$\Longrightarrow \boxed{J_n(\mu b) = 0}$$

$$\Longrightarrow \mu b = z_1, z_2, z_3, \cdots z_k, \cdots \quad \boxed{\text{zeros of } J_n(\cdot)}$$

$$\Longrightarrow \mu = \frac{z_k}{b}, \qquad \lambda_k = -\mu^2 = -\left(\frac{z_k}{b}\right)^2$$

$\Delta u = \lambda u$ in $D_b = \{ x^2 + y^2 \le b^2 \}$

$0 < r < b$

$0 < \theta < 2\pi$

② $R(r) = A J_n(\mu r)$

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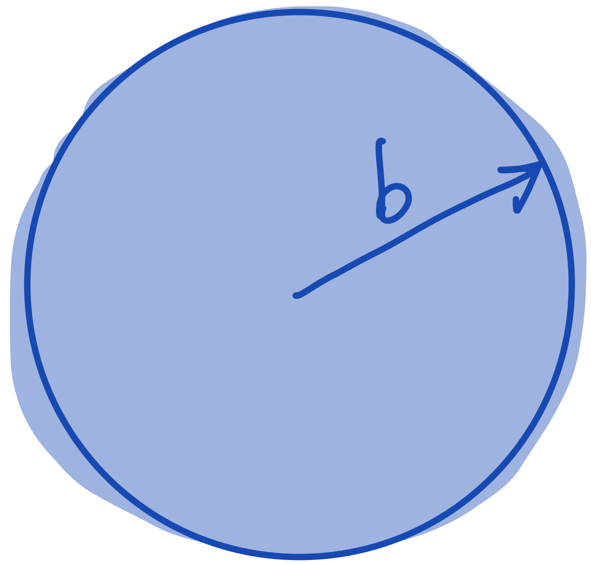



Boundary condition at $r = b$:

Neumann B.C. $\quad R'(b) = 0$

$$R'(r) = A \mu J_n'(\mu r)$$

$$R'(b) = 0 \Longrightarrow \mu J_n'(\mu b) = 0$$

$$\Longrightarrow (\mu b) J_n'(\mu b) = 0 \quad \text{i.e.} \quad \boxed{x J_n'(x) = 0}$$

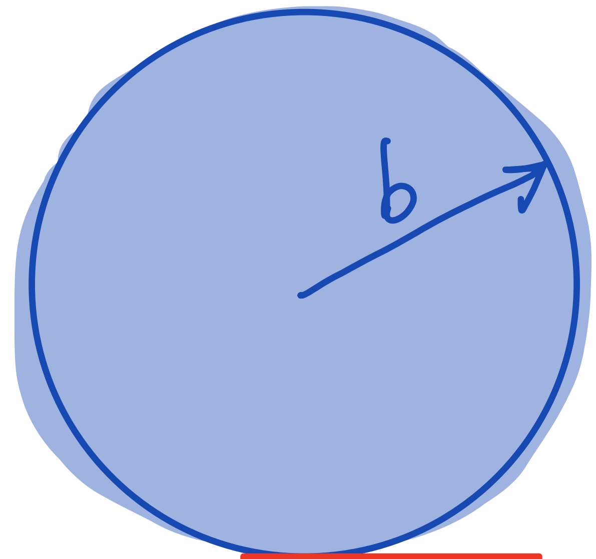
$$\Delta u = \lambda u \quad \text{in } D_b = \{x^2+y^2 < b^2\}$$

$0 < r < b$

$0 < \theta < 2\pi$

② $R(r) = A J_n(\mu r)$

$(\lambda = -\mu^2)$




Boundary condition at $r=b$:

Neumann B.C. $\quad R'(b) = 0$

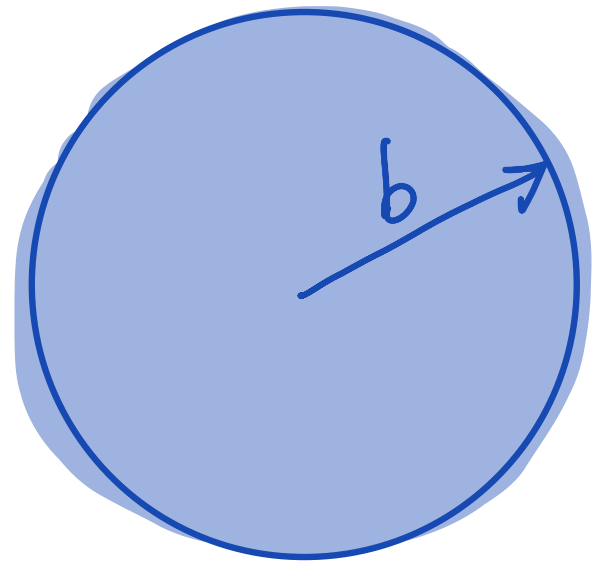
$\Longrightarrow \mu b = z_1, z_2, \dots, z_k, \dots$ zeros of $\boxed{x J_n'(x)}$

$\Longrightarrow \mu = \frac{z_k}{b}, \quad \lambda_k = -\mu^2 = -\left(\frac{z_k}{b}\right)^2$

$\Delta u = \lambda u$ in $D_b = \{x^2+y^2 \leq b^2\}$

$0 < r < b$

$0 \leq \theta < 2\pi$




② $R(r) = A J_n(\mu r)$

$(\lambda = -\mu^2)$

Boundary condition at $r=b$:

Robins B.C. $R'(b) + \beta R(b) = 0$

$(\beta > 0)$

$$R'(r) + \beta R(r) = A\left[\mu J_n'(\mu r) + \beta J_n(\mu r)\right]$$

$$R'(b) + \beta R(b) = A\left[\mu J_n'(\mu b) + \beta J_n(\mu b)\right] = 0$$

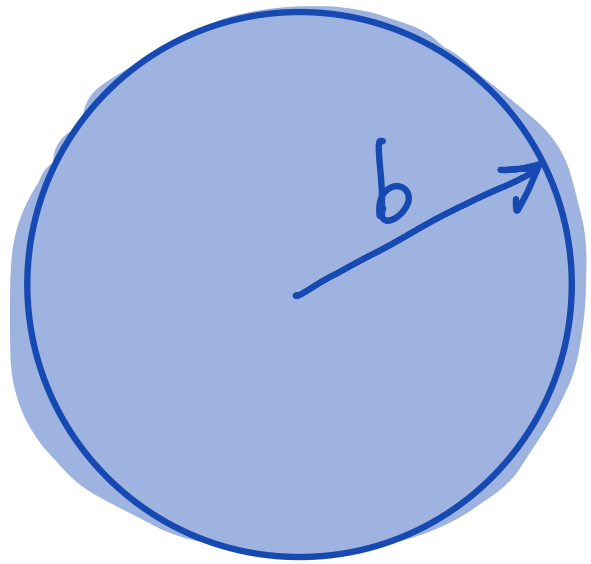
## $\Delta u = \lambda u$ in $D_b = \{x^2+y^2 < b^2\}$

$0 < r < b$

$0 < \theta < 2\pi$

② $R(r) = A J_n(\mu r)$

$(\lambda = -\mu^2)$




**Boundary condition at $r=b$:**

Robins B.C. $R'(b) + \beta R(b) = 0$

$(\beta > 0)$

$$\mu J_n'(\mu b) + \beta J_n(\mu b) = \frac{1}{b}\mu b J_n'(\mu b) + \beta J_n(\mu b) = 0$$

Let $z_1, z_2, \dots, z_k$ be zeros of $\boxed{\frac{1}{b} x J_n'(x) + \beta J_n(x) = 0}$

$$\Delta u = \lambda u \text{ in } D_b = \{x^2+y^2 < b^2\}$$

$0 < r < b$

$0 < \theta < 2\pi$

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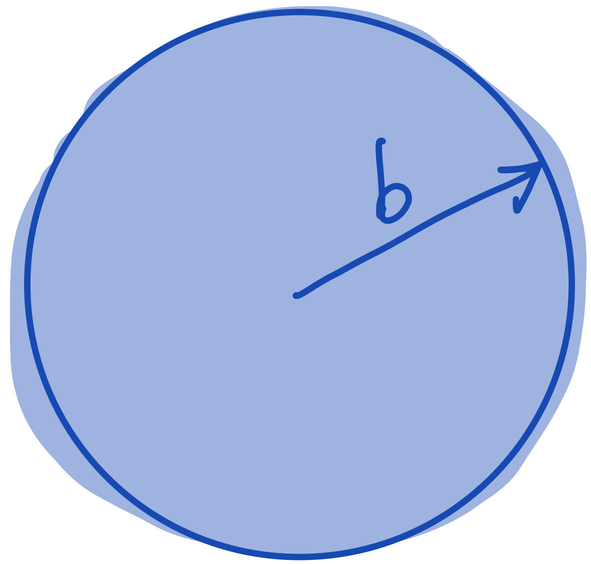



Let $z_1, z_2, \dots, z_k$ be zeros of $\boxed{\frac{1}{b} x J_n'(x) + \beta J_n(x) = 0}$

$$\mu = \frac{z_k}{b}, \qquad \lambda = -\mu^2 = -\left(\frac{z_k}{b}\right)^2$$

$$\Delta u = \lambda u \quad \text{in } D_b = \{x^2+y^2 \le b^2\}$$

**Eigenfunctions:**

$$\boxed{u(r,\theta) = R_n(r)\,\Phi_n(\theta)}$$

$\Phi_n(\theta) = 1, \cos n\theta, \sin n\theta$
$(n=0)$

$R_n(r) = J_n(\mu r) = J_n\left(\frac{z_k}{b} r\right)$

$\mu b = z_{k,n} \quad \mu = \frac{z_{k,n}}{b}$

$$= \boxed{J_n\left(\frac{z_{k,n}\, r}{b}\right)\Phi_n(\theta)}, \quad n = 0, 1, 2, 3, \ldots, \quad k = 1, 2, 3, \ldots$$

Note: $z_k = z_{k,n}$, as they are zeros related to $J_n$

$$\Delta u = \lambda u \quad \text{in} \quad \Omega = \{a \le r \le b,\ 0 \le \theta \le \alpha\}$$

## Eigenfunctions:

$$\boxed{u(r,\theta) = R_n(r)\,\Phi_n(\theta)}$$

$\Phi_n(\theta) = \Phi_{\nu_n}(\theta)$,

$\nu_n$'s are determined by B.C. of $\theta$

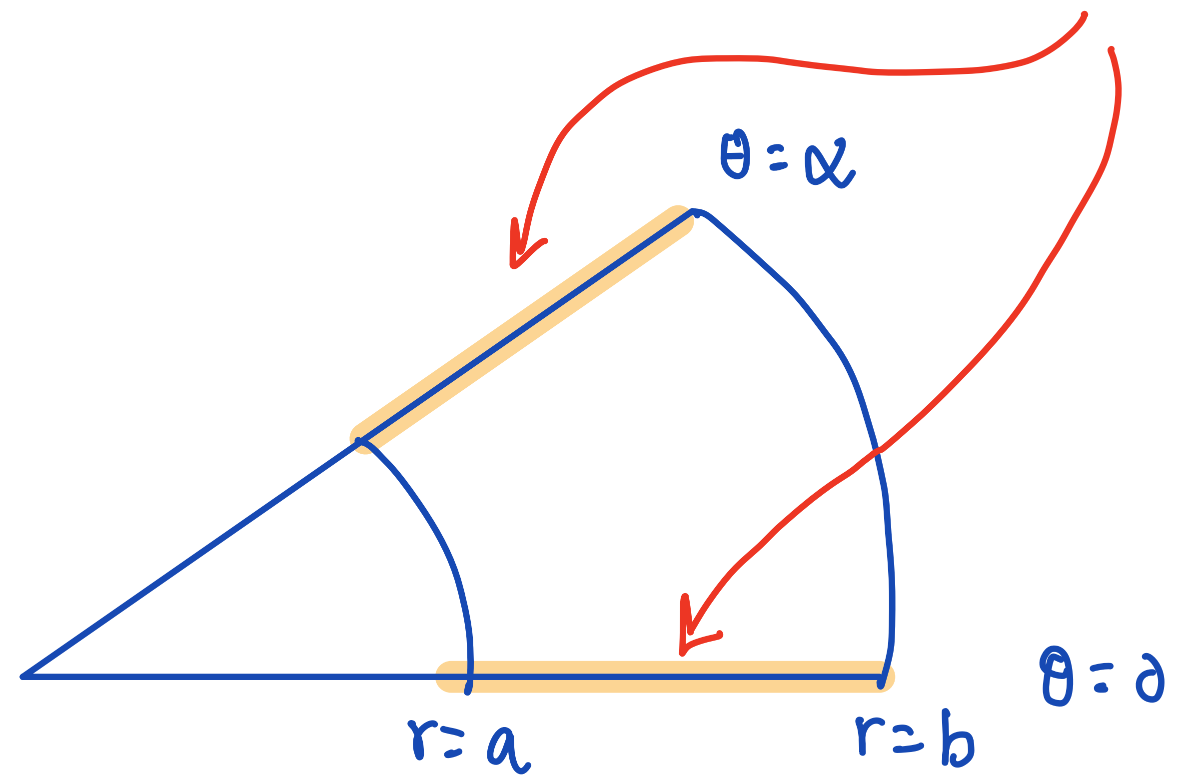
$$R_n(r) = J_{\nu_n}(\mu r) = J_{\nu_n}\left(\frac{z_{k,n}}{b} r\right)$$

$$\mu b = z_{k,n} \quad \mu = \frac{z_{k,n}}{b}$$

$$= \boxed{J_n\left(\frac{z_{k,n}\, r}{b}\right)\Phi_n(\theta)}, \quad n = 0, 1, 2, 3, \cdots, \quad k = 1, 2, 3, \cdots$$

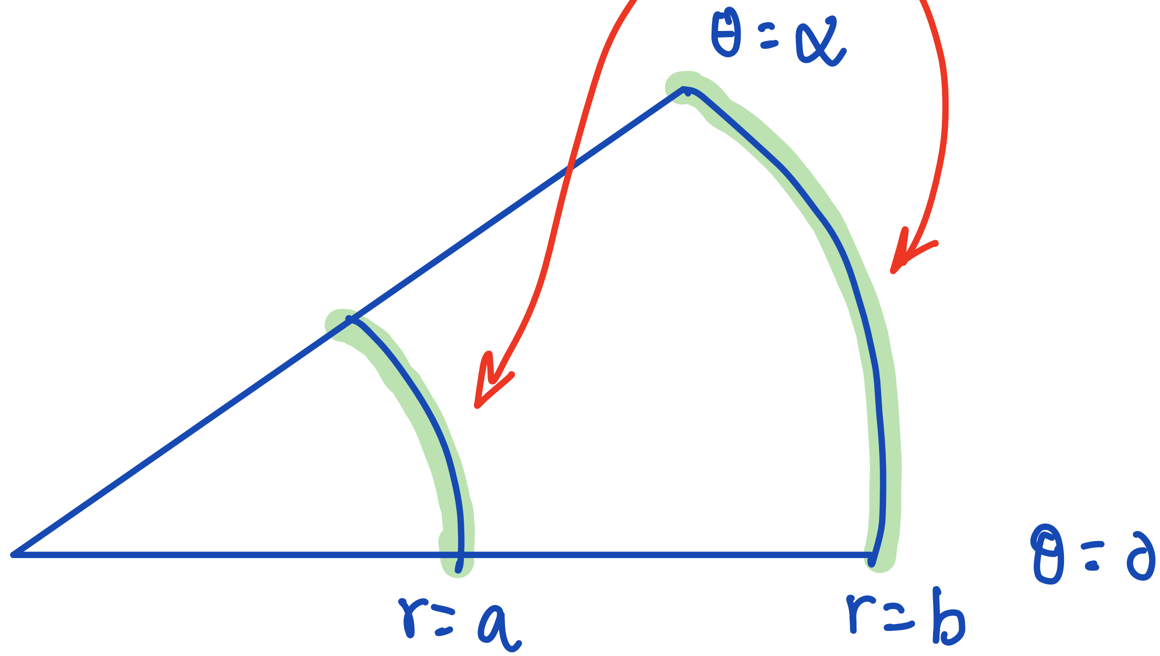
Note: $z_k = z_{k,n}$, as they are zeros related to $J_{\nu_n}$

$$\Delta u = \lambda u \quad \text{in} \quad \Omega = \{a \le r \le b,\ 0 \le \theta \le \alpha\}$$




$$\boxed{\begin{cases} \Phi''(\theta) + \nu \Phi(\theta) = 0 \\ \text{B.C. at } \theta = 0, \alpha \end{cases} \Longrightarrow \nu = \nu_n \cdots}$$

$$\Delta u = \lambda u \quad \text{in} \quad \Omega = \{a \le r \le b,\ 0 \le \theta \le \alpha\}$$




$$\boxed{\begin{array}{l} r^2 R''(r) + r R'(r) + (\mu^2 r^2 - \nu_n^2) R(r) = 0 \\ \text{B.C. at } r = a, b \Longrightarrow \mu = \mu_n = z_{k,n} \cdots \end{array}}$$

# Orthogonality of Eigenfunctions of $\Delta$

$$\left\{ u(r,\theta) = u_{n,k}(r,\theta) = R_k(r)\Phi_n(\theta) \right\}_{n,k}$$

$$\boxed{\begin{aligned}
&\langle u_{n,k}, u_{m,\ell} \rangle_{L^2(x,y)} = 0 \quad \text{if } (n,k) \neq (m,\ell) \\
&\iint R_k(r)\Phi_n(\theta) R_\ell(r)\Phi_m(\theta)\, dx\, dy \\
&= \iint R_k(r)\Phi_n(\theta) R_\ell(r)\Phi_m(\theta)\, r\, dr\, d\theta = 0 \\
&= \Big( \int \underbrace{\Phi_n(\theta)\Phi_m(\theta)\, d\theta}_{=0 \text{ if } n\neq m} \Big) \Big( \int \underbrace{R_k(r)R_\ell(r)\, r\, dr}_{=0 \text{ if } n=m,\ k\neq\ell} \Big) = 0
\end{aligned}}$$

$\perp$ in $\langle\ \rangle_{L^2(x,y)}$

# Orthogonality of Eigenfunctions of $\Delta$

$$\left\{ u(r,\theta) = u_{n,k}(r,\theta) = R_k(r)\,\Phi_n(\theta) \right\}_{n,k}$$

$$\boxed{f(r,\theta) = \sum_{n,k} a_{n,k}\, u_{n,k}(r,\theta)}$$

"Fourier Series" expansion

$$\boxed{a_{n,k} = \frac{\langle f, u_{n,k} \rangle}{\langle u_{n,k}, u_{n,k} \rangle}}$$

# Orthogonality of Eigenfunctions of $\Delta$

$$\left\{ u(r,\theta) = u_{n,k}(r,\theta) = R_k(r)\,\Phi_n(\theta) \right\}_{n,k}$$

$$f(r,\theta) = \sum_{n,k} a_{n,k}\, u_{n,k}(r,\theta)$$

$$\iint f(r,\theta)\, u_{n,k}(r,\theta)\, r\,dr\,d\theta$$

$$a_{n,k} = \frac{\langle f, u_{n,k} \rangle}{\langle u_{n,k}, u_{n,k} \rangle}$$

# Orthogonality of Eigenfunctions of $\Delta$

$$\left\{ U(r,\theta) = U_{n,k}(r,\theta) = R_k(r)\,\Phi_n(\theta) \right\}_{n,k}$$

$$f(r,\theta) = \sum_{n,k} a_{n,k}\, U_{n,k}(r,\theta)$$

$$a_{n,k} = \frac{\langle f, U_{n,k} \rangle}{\langle U_{n,k}, U_{n,k} \rangle}$$

$$\iint U_{n,k}^2(r,\theta)\, r\, dr\, d\theta$$

$$= \iint R_k^2(r)\, \Phi_n^2(\theta)\, r\, dr\, d\theta$$

$$= \left( \int \Phi_n^2(\theta)\, d\theta \right) \left( \underbrace{\int R_k^2(r)\, r\, dr}_{\text{(Lemma 5.4)}} \right)$$

Lemma 5.4 [F, p.146]

$\forall\, \mu>0,\ b>0,\ \nu\geq 0,$

$$\boxed{\int_0^b J_\nu(\mu r)^2\, r\, dr = \frac{b^2}{2} J_\nu'(\mu b)^2 + \frac{\mu^2 b^2 - \nu^2}{2\mu^2} J_\nu(\mu b)^2}$$

Pf.

$$r^2 R''(r) + rR'(r) + (\mu^2 r^2 - \nu^2) R(r) = 0$$

$x = \mu r, \quad R(r) = f(x) = f(\mu r)$

$$x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0$$

$$x\left[x f''(x) + f'(x)\right] + (x^2 - \nu^2) f(x) = 0$$

$$x\left[x f'(x)\right]' + (x^2 - \nu^2) f(x) = 0$$

$f'(x) \rightarrow$

$$x\left[x f'(x)\right]' f'(x) + (x^2 - \nu^2) f(x) f'(x) = 0$$

$$\left[x f'(x)\right]'\left[x f'(x)\right] + (x^2 - \nu^2) f(x) f'(x) = 0$$

$$\cancel{\frac{1}{2}} \frac{d}{dx}\left[(x f')^2\right] + \cancel{\frac{1}{2}} (x^2 - \nu^2) \frac{d}{dx}(f^2) = 0$$

$$\int_0^\ell \Rightarrow \int_0^\ell \frac{d}{dx}(xf')^2\,dx + \int_0^\ell (x^2-\nu^2)\frac{d}{dx}(f^2)\,dx = 0$$

$$(x f'(x))^2\Big|_0^\ell + (x^2-\nu^2)f^2\Big|_0^\ell - 2\int_0^\ell x f^2\,dx = 0$$

$$\boxed{\int_0^\ell x f^2(x)\,dx = \frac{\ell^2 f'(\ell)^2}{2} + \frac{(\ell^2-\nu^2)f^2(\ell)}{2} + \frac{\nu^2 f^2(0)}{2}}$$

Now take $f(x) = J_\nu(x)$.

- <u>If $\nu > 0$, then $f(0) = J_\nu(0) = 0$</u>

$$\left( J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu} \qquad \text{F. p. 130 (5.10)} \right)$$

- <u>If $\nu = 0$, then $\nu^2 f^2(0) = 0$</u>

Hence

$$\boxed{\int_0^\ell J_\nu^2(x)\,x\,dx = \frac{\ell^2 J_\nu'(\ell)^2}{2} + \frac{(\ell^2-\nu^2)J_\nu(\ell)^2}{2}}$$

Now, recall $x = \mu r$. Let also $l = \mu b$. Then

$$\int_0^{x=\mu b} J_\nu^2(\mu r)\mu r \, d(\mu r)$$

$$= \frac{\mu^2 b^2 J_\nu'(\mu b)^2}{2} + \frac{(\mu b)^2 - \nu^2}{2} J_\nu(\mu b)^2$$

$$\boxed{\int_0^b J_\nu^2(\mu r) r \, dr = \frac{b^2}{2} J_\nu'(\mu b)^2 + \frac{(\mu b)^2 - \nu^2}{2\mu^2} J_\nu(\mu b)^2}$$