Orthogonal Polynomials (F. Chapter 6)

(6.2) Legendre Polynomials

$$\boxed{P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right]}$$

$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$,

$P_3(x) = \frac{1}{2}(5x^3-3x)$, $P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[x^{2n} + \cdots\cdots\right]$$

$$= \frac{1}{2^n n!}\left[(2n)(2n-1)\cdots(2n+1)\, x^{2n} + \cdots\cdots\right]$$

$$= \frac{(2n)!}{2^n (n!)^2} x^n + \cdots\cdots$$

Rodrigues formula: $\boxed{p_n(x) = \frac{C_n}{w(x)} \frac{d^n}{dx^n}\left(w(x)P(x)^n\right)}$

$w(x) \equiv 1$ $\qquad$ $P(x) \equiv x^2-1$

Thm 6.1 $\{P_n\}_{n=0}^{\infty}$ — orthogonal in $L^2(-1,1)$

$$\|P_n\|^2 = \frac{2}{2n+1}$$

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$$

Thm 6.3 $\{P_n\}_{n=0}^{\infty}$ — orthogonal basis in $L^2(-1,1)$

Thm 6.4 $\{P_{2n}\}_{n=0}^{\infty}$, $\{P_{2n+1}\}_{n=0}^{\infty}$ are bases for $L^2(0,1)$, $\|P_k\|^2 = \frac{1}{2k+1}$

$\lambda = -n(n+1)$

Thm 6.2 $\forall n \geq 0$,

$$\boxed{\left[(1-x^2)P_n'(x)\right]' + n(n+1)P_n(x) = 0}$$

(Sturm-Liouville Operator:

weight

$$(r(x)f')' + p(x)f = \lambda w(x) f$$)

Thm 6.6 (Recurrence formula)

$$P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x)$$

(F. p.173: #5

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0.$$)

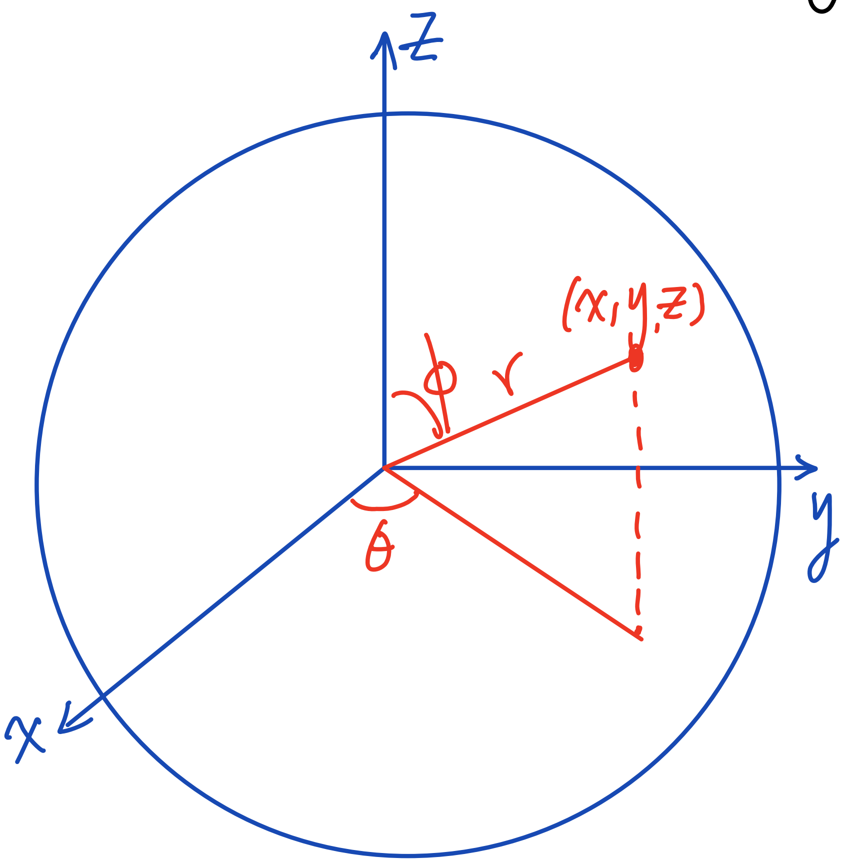
Recall: e^x : $f'(x) = f(x)$

$\cos x, \sin x$: $f''(x) + f(x) = 0$

$J_\nu(x)$: $x^2 f''(x) + x f'(x) + (x^2 - \nu^2) f(x) = 0$

$I_\nu(x)$: $x^2 f''(x) + x f'(x) - (x^2 + \nu^2) f(x) = 0$

Spherical Coordinates & Legendre fcts



$$\begin{cases} x = r\sin\phi\cos\theta \\ y = r\sin\phi\sin\theta \\ z = r\cos\phi \end{cases}$$

$$\left(\begin{array}{c} 0 < r, \\ 0 < \theta < 2\pi \\ 0 < \phi < \pi \end{array} \right)$$

$$u = u(r, \theta, \phi)$$

$$\boxed{\Delta u = u_{rr} + \frac{2}{r} u_r + \frac{1}{r^2 \sin\phi} \partial_\phi (u_\phi \sin\phi) + \frac{1}{r^2 \sin^2\phi} \partial_\theta^2 u}$$

Dirichlet Problem in a Ball

$$\begin{cases} \Delta u = 0 \quad \text{on} \quad B_1 = \{(x,y,z) : \underbrace{x^2 + y^2 + z^2 < 1}_{r<1}\} \\ u(1, \theta, \phi) = f(\theta, \phi) \end{cases}$$

$$u(r,\theta,\phi) = R(r)\,\Theta(\theta)\,\Phi(\phi)$$

$0 < r < 1 \qquad 0 < \theta < 2\pi \qquad 0 < \phi < \pi$

$$\boxed{(r^2\sin^2\phi)\left[\frac{R''}{R} + \frac{2R'}{rR}\right] + (\sin\phi)\frac{(\Phi'\sin\phi)'}{\Phi} + \frac{\Theta''}{\Theta} = 0}$$

$$\underbrace{(r^2\sin^2\phi)\left[\frac{R''}{R} + \frac{2R'}{rR}\right] + (\sin\phi)\frac{(\Phi'\sin\phi)'}{\Phi}}_{} = \underbrace{-\frac{\Theta''}{\Theta} = m^2}_{}$$

① Θ -2π-per in θ, $\Theta'' + m^2\Theta = 0$

$$\boxed{\Theta = \Theta_m = e^{im\theta},\ e^{-im\theta}}$$

$$\frac{r^2R'' + 2rR'}{R} = \frac{m^2}{\sin^2\phi} - \frac{(\Phi'\sin\phi)'}{\Phi\sin\phi} = \lambda$$

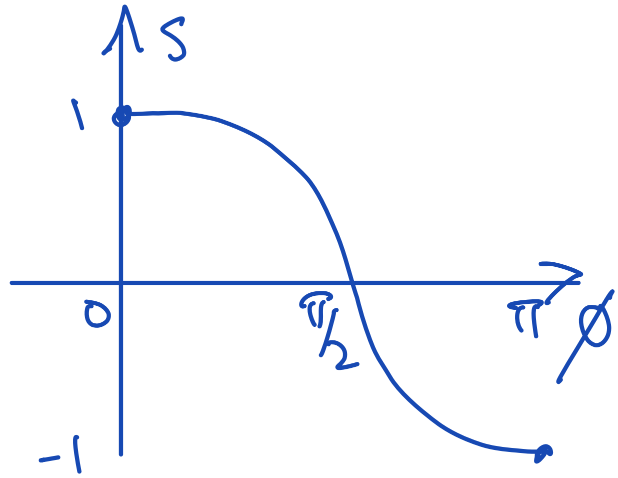
② $$\frac{(\Phi'\sin\phi)'}{\sin\phi} - \frac{m^2\Phi}{\sin^2\phi} + \lambda\Phi = 0 \qquad (6.15)$$

③ $$r^2R'' + 2rR' - \lambda R = 0 \qquad (16.16)$$

For (6.15), use change of variable:

$s = \cos\phi, \quad \phi = \cos^{-1} s,$

$0 < \phi < \pi, \quad -1 < s < 1$



Let $\Phi(\phi) = Q(s)$

$$\Phi'(\phi) = Q'(s)\frac{ds}{d\phi} = -(\sin\phi)\,Q'(s)$$

$$\frac{\Phi'(\phi)}{\sin\phi} = -Q'(s), \qquad \boxed{\frac{1}{\sin\phi}\frac{d}{d\phi} = \frac{d}{ds}}$$

$$\boxed{\frac{(\Phi'\sin\phi)'}{\sin\phi} - \frac{m^2\Phi}{\sin^2\phi} + \lambda\Phi = 0}$$

$$\boxed{-\frac{d}{ds}\left((\sin^2\phi)(-Q'(s))\right) - \frac{m^2 Q(s)}{\sin^2\phi} + \lambda Q(s) = 0}$$

(6.17)

$$\boxed{\frac{d}{ds}\left[(1-s^2)\frac{d}{ds}Q(s)\right]-\frac{m^2Q(s)}{1-s^2}+\lambda Q(s)=0}$$

(Legendre Eqn of order m)

m = positive integer, $Q(s)=(1-s^2)^{m/2}\frac{d^m}{ds^m}W(s)$

$$\boxed{\left((1-s^2)W'(s)\right)'+\lambda W(s)=0}$$

Legendre poly

Take $\lambda=n(n+1)$, $W(s)=P_n(s)=\frac{1}{2^n n!}\frac{d^n}{ds^n}\left((s^2-1)^n\right)$

$$\boxed{Q(s)=\underbrace{\frac{(1-s^2)^{m/2}}{2^n n!}\frac{d^{m+n}}{ds^{m+n}}\left[(s^2-1)^n\right]}_{P_n^m(s)},\quad n\geq m}$$

Thm 6.7 For each m, positive integer, $\{P_n^m(s)\}_{n\geq m}$ is an orthogonal basis for $L^2(-1,1)$

$$\|P_n^m\|^2=\frac{(n+m)!}{(n-m)!}\frac{2}{2n+1}$$

For (6.16),

$$\boxed{r^2R''(r) + 2rR'(r) - n(n+1)R(r) = 0}$$

Euler equation: Set $R(r) = r^\lambda$

$$\lambda(\lambda-1) + 2\lambda - n(n+1) = 0$$

$$\lambda^2 + \lambda - n(n+1) = 0$$

$$(\lambda - n)(\lambda + n + 1) = 0$$

$$\lambda = n, -n-1.$$

$R(0^+)$ exists, finite

$$R(r) = Ar^n + \cancel{Br^{-n-1}} \qquad 0 < r < 1$$

$$\boxed{R(r) = Ar^n.}$$

$$\boxed{u_{mn}(r,\theta,\phi) = r^n e^{im\theta} P_n^m(\cos\phi) \qquad |m| \leq n, \quad n = 0, 1, 2, 3, \dots}$$

Solution of Dirichlet Problem

$$\boxed{u(r,\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} C_{mn} r^n e^{im\theta} P_n^m(\cos\phi)}$$

$$(r=1) \quad f(\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} C_{mn} e^{im\theta} P_n^m(\cos\phi)$$

$$C_{mn} = \frac{\langle f, Y_{mn} \rangle}{\|Y_{mn}\|^2}$$

$$= \frac{(2n+1)(n-|m|)!}{4\pi(n+|m|)!} \int_0^{\pi} \int_0^{2\pi} f(\theta,\phi) e^{-im\theta} P_n^m(\cos\phi) \sin\phi \, d\theta \, d\phi$$

Spherical harmonics

$$Y_{mn}(\theta,\phi) = e^{im\theta} P_n^m(\cos\phi)$$

surface measure

$$\|Y_{mn}\|^2 = \int_0^{2\pi} \int_0^{\pi} Y_{mn}(\theta,\phi)^2 \sin\phi \, d\phi \, d\theta$$

$$= \frac{4\pi}{2n+1} \frac{(n+|m|)!}{(n-|m|)!}$$

Thm 6.10 ("Poisson Kernel" for the Ball) (6.27)

$$u(x) = \frac{1}{4\pi} \iint\limits_{|y|=1} \frac{1-|x|^2}{\left(1-2|x|\cos\alpha + |x|^2\right)^{3/2}} f(y)\, d\sigma(y)$$

(α = angle between $\vec{x}$ & $\vec{y}$

$\cos\alpha = \dfrac{\langle x, y\rangle}{\|x\|\,\|y\|}$)

(Surface measure on $|y|=1$)

Compare with Poisson Kernel for the disk (4.35)

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)} f(\phi)\, d\phi$$

Eigenvalues of the Laplacian in a Ball

$\{|x| \le 1\}$

$$\begin{cases} \Delta u = \lambda u & (\lambda = -\mu^2) \\ u\big|_{|x|=1} = 0 \end{cases}$$

$$u(r,\theta,\phi) = R(r)\,\Theta(\theta)\,\Phi(\phi)$$

① $\Theta(\theta) = e^{im\theta},\ e^{-im\theta}$

② $\Phi(\phi) = P_n^m(\cos\phi)$

(6.31)

③ $r^2 R''(r) + 2rR'(r) + (\mu^2 r^2 - n(n+1))R(r) = 0$

Let $R(r) = r^{-1/2} g(r)$

Then $g(r)$ solves:

$$r^2 g'' + r g' + \left[\mu^2 r^2 - (n+\tfrac{1}{2})^2\right] g(r) = 0$$

$$g(r) = J_{n+\frac{1}{2}}(\mu r)$$

$$R(r) = r^{-1/2} J_{n+\frac{1}{2}}(\mu r)$$

(6.32)

$$\boxed{U_{m,n,k}(r,\theta,\phi) = r^{-1/2} J_{n+\frac{1}{2}}(\mu_k^n r)\, e^{im\theta} P_n^{|m|}(\cos\phi)}$$

where μ_k^n satisfies (e.g. $R(1)=0$: Dir. B.C.)

$$J_{n+\frac{1}{2}}(\mu_k^n)=0, \quad k=1,2,3,\cdots$$

$$\|U_{m,n,k}(r,\theta,\phi)\|^2$$

$dx\,dy\,dz = r^2 \sin\phi \, dr\, d\theta\, d\phi$

$$= \int_0^1 \int_0^{2\pi} \int_0^{\pi} |U_{m,n,k}(r,\theta,\phi)|^2 \, r^2 \sin\phi \, dr\, d\theta\, d\phi$$

$$= \frac{2\pi (n+|m|)!\, J_{n+\frac{3}{2}}^2(\mu_k^n)}{(2n+1)(n-|m|)!}$$

(P.183)

Hermite Polynomials

$w(x) = e^{-x^2}$, $p(x) = 1$

$$\boxed{H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)}$$

$H_0(x) = 1$; $H_1(x) = 2x$; $H_2(x) = 4x^2 - 2$

$H_3(x) = 8x^3 - 12x$, $H_4(x) = 16x^4 - 48x^2 + 12$

Thm 6.11. $\{H_n\}_{n=0}^{\infty}$ are orthogonal in $L^2_w(\mathbb{R})$ with weight $w(x) = e^{-x^2}$ and

$$\|H_n\|_w^2 = 2^n n! \sqrt{\pi}$$

$$\left(\quad \langle f, g \rangle_w = \int_{-\infty}^{\infty} f(x) g(x) e^{-x^2} dx \right.$$

$$\left. \int_{-\infty}^{\infty} \left(e^{x^2} \frac{d^m}{dx^m} e^{-x^2}\right) \left(e^{x^2} \frac{d^n}{dx^n} e^{-x^2}\right) e^{-x^2} dx = 0 \quad \text{if } m \neq n \right)$$

Thm 6.12 $\{H_n\}_{n=0}^{\infty}$ is an orthogonal basis for $L^2_w(\mathbb{R})$.

Recurrence formula

$$H_n(x) = 2x H_{n-1}(x) - H_{n-1}'(x) \qquad (6.34)$$

$$H_0'(x) = 0,$$

$$H_n' = 2n H_{n-1}$$

$$H_n(x) = 2x H_{n-1}(x) - 2(n-1) H_{n-2}(x) \qquad (6.37)$$

$$\boxed{\begin{gathered} H_n''(x) - 2x H_n'(x) + 2n H_n(x) = 0 \\ \left(e^{-x^2} H_n'\right)' + 2n e^{-x^2} H_n = 0 \\ \left(\left(e^{-x^2} y'(x)\right)' + \lambda e^{-x^2} y(x) = 0 \right) \end{gathered}}$$

Hermite function: $h_n(x) = e^{-x^2/2} H_n(x)$ $\quad (\omega = 1)$

Thm 6.14 $\{h_n\}_{n=0}^{\infty}$ — orthogonal basis for $L^2(\mathbb{R})$,

$$x h_n(x) + h_n'(x) = 2n h_{n-1}(x)$$

$$x h_n(x) - h_n'(x) = h_{n+1}(x)$$

$$\boxed{h_n''(x) - x^2 h_n(x) + (2n+1) h_n(x) = 0}$$

Application in Quantum Mechanics

1D Schrodinger equation

(1) $$-iu_t = u_{xx} - x^2 u$$

Complex i potential

$\boxed{\text{(Quantum harmonic oscillator)}}$

(2) Eigenvalue and eigenfunction:

$$v_{xx} - x^2 v = -\lambda v$$

energy

$$\Longleftrightarrow \quad \boxed{v_{xx} + (\lambda - x^2) v = 0}$$

(3) Let $v(x) = e^{-x^2/2} w(x)$

$\Longrightarrow$ $w(x)$ solves: $\lambda - 1 = 2n$, i.e. $\lambda = 2n+1$.

$$\boxed{w''(x) - 2x w'(x) + (\lambda - 1) w(x) = 0}$$

(cf. $H_n''(x) - 2x H_n'(x) + 2n H_n(x) = 0$)

$$\boxed{v(x) = e^{-\frac{x^2}{2}} H_n(x) = h_n(x), \quad \lambda = 2n+1}$$

Laguerre Polynomials

$w(x) = x^{\alpha} e^{-x}, \; P(x) = x$

$$\boxed{L_n^{\alpha}(x) = \frac{x^{-\alpha} e^{x}}{n!} \frac{d^n}{dx^n}\left(x^{\alpha+n} e^{-x}\right)}, \quad \alpha > -1$$

Thm 6.15 $\{L_n^{\alpha}\}_{n=0}^{\infty}$ is a complete orthogonal basis for $L_w^2(0,\infty)$ with weight $w(x) = x^{\alpha} e^{-x}$,

and
$$\| L_n^{\alpha} \|_w^2 = \frac{\Gamma(n+\alpha+1)}{n!}$$

Thm 6.16: L_n^{α} satisfies:

$$\boxed{x y''(x) + (\alpha+1-x) y' + n y = 0}$$

$\Updownarrow$

$$\boxed{\left(x^{\alpha+1} e^{-x} y'\right)' + n x^{\alpha} e^{-x} y = 0}$$

Application in Quantum Mechanics

(hydrogen atom)

$$-i u_t = \frac{1}{2}\Delta u + \frac{1}{r} u$$

potential function

Eigenvalue and eigenfunction:

$$\Delta v + \frac{2}{r} v = -\lambda v$$

$$v(x) = R(r)\,\underbrace{\Theta(\theta)}_{e^{im\theta}}\,\underbrace{\Phi(\phi)}_{P_n^{|m|}(\cos\phi),\ n \geq |m|}$$

$$\boxed{r^2 R'' + 2rR' + \left(2\lambda r^2 + 2r - n(n+1)\right) R = 0}$$

(6.53)

$$\nu = (-2\lambda)^{1/2}, \quad s = 2\nu^{-1} r,$$

$$R(r) = S(s) = S(2\nu^{-1} r)$$

(6.54)

$$\boxed{s^2 S'' + 2s S' + \left(\nu s - \frac{1}{4} s^2 - n(n+1)\right) S = 0}$$

$$S(s) = s^{n} e^{-\frac{s}{2}} \Sigma(s)$$

$$\boxed{s\Sigma'' + (2n+2-s)\Sigma' + (\nu-n-1)\Sigma = 0}$$

(cf. $xy''(x) + (\alpha+1-x)y' + ly = 0$, $y = L_{l}^{\alpha}$)

$$\alpha+1 = 2n+2 \Rightarrow \alpha = 2n+1, \quad \nu = l+n+1$$

$$\Sigma(s) = L_{l}^{2n+1}(s), \quad \underline{\nu = l+n+1}$$

$$S(s) = s^{\frac{n}{2}} e^{-\frac{s}{2}} L_{l}^{2n+1}(s)$$

$$R(r) = (2\nu^{-1}r)^{\frac{n}{2}} e^{-\frac{r}{\nu}} L_{l}^{2n+1}(2\nu^{-1}r) \tag{6.55}$$

$$\boxed{v(x) = R(r)\, e^{im\theta} P_{n}^{|m|}(\cos\phi)} \tag{6.56}$$