Fourier Transform

$f(x): \ -\infty < x < \infty$

$$\hat{f}(\xi) = (\mathcal{F}f)(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx$$

$$f(x) = (\mathcal{F}^{-1}\hat{f})(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x}\, d\xi$$

$f(x): \ -L < x < L, \ 2L$-periodic

$$C_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-\frac{in\pi x}{L}}\, dx$$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{L}}$$

Fourier Transform

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$$f(x) = \underline{\sum_{n=-\infty}^{\infty}} C_n e^{\frac{in\pi x}{L}}$$

ξ

n

Solution of Heat Equation

$$u_t = D u_{xx} \qquad -\pi < x < \pi,$$

$$u(-\pi,t) = u(\pi,t) \qquad \text{periodic B.C.}$$

$$u(x,0) = f(x) \qquad \text{initial data}$$

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t)\cos nx + b_n(t)\sin nx$$

$$= \frac{a_0(0)}{2} + \sum_{n=1}^{\infty} a_n(0)e^{-Dn^2t}\cos nx + b_n(0)e^{-Dn^2t}\sin nx$$

$$a_n(0) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n(0) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Solution of Heat Equation

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Solution of Heat Equation

$$\begin{tikzcd}
u(x,0) \arrow[r] \arrow[d, "F.S."] & u_t = D u_{xx} \arrow[r] & u(x,t) \\
C_n(0) \arrow[r] & e^{-Dn^2 t} \arrow[r] & C_n(0)\, e^{-Dn^2 t} \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
u(x,0) \arrow[r] \arrow[d, swap, "\mathcal{F}"] \arrow[d, "F.T."] & u_t = D u_{xx} \arrow[r] & u(x,t) \\
\hat{u}(\xi,0) \arrow[r] & e^{-D\xi^2 t} \arrow[r] & \hat{u}(\xi,0)\, e^{-D\xi^2 t} \arrow[u, swap, "\mathcal{F}^{-1}"]
\end{tikzcd}$$

Solution of Heat Equation

$$
\begin{tikzcd}
u(x,0) \arrow[r] \arrow[d, "F.S."] & u_t = D u_{xx} \arrow[r] & u(x,t) \\
C_n(0) \arrow[r] & e^{-Dn^2 t} \arrow[r] & C_n(0)\, e^{-Dn^2 t} \arrow[u]
\end{tikzcd}
$$

$$
\begin{tikzcd}
u(x,0) \arrow[r] \arrow[d, swap, "F"] \arrow[d, "F.T."] & u_t = D u_{xx} \arrow[r] & u(x,t) \\
\hat{u}(\xi,0) \arrow[r] & e^{-D\xi^2 t} \arrow[r] & \hat{u}(\xi,0)\, e^{-D\xi^2 t} \arrow[u, swap, "F^{-1}"]
\end{tikzcd}
$$

Convolution in x = Multiplication in ξ

$$(f*g)(x) = \int_{-\infty}^{\infty} f(y)\,g(x-y)\,dy$$

$$= \int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy = (g*f)(x)$$

$$\boxed{\widehat{f*g}(\xi) = \hat{f}(\xi)\,\hat{g}(\xi)}$$

Convolution in x = Multiplication in ξ

Back to heat equation:

$$\hat{u}(\xi,t) = \hat{u}(\xi,0)\, e^{-D\xi^2 t}$$

$$u(x,0)\ (f(x)) \longrightarrow \hat{u}(\xi,0)\ (\hat{f}(\xi))$$

$$g(x,t) \longrightarrow e^{-D\xi^2 t}$$

$$u(x,t) = (f * g)(x,t) = \int_{-\infty}^{\infty} f(y)\, g(x-y,t)\, dy$$

Convolution in x = Multiplication in ξ

Back to heat equation:

$$\hat{u}(\xi,t) = \hat{u}(\xi,0)\, e^{-D\xi^2 t}$$

$$u(x,0)\ (f(x)) \longrightarrow \hat{u}(\xi,0)\ (\hat{f}(\xi))$$

$$g(x,t) \longrightarrow e^{-D\xi^2 t}$$ ← Heat Kernel

$$\boxed{u(x,t) = (f*g)(x,t) = \int_{-\infty}^{\infty} f(y)\, g(x-y,t)\, dy}$$

Duality between $\mathcal{F}$ & $\mathcal{F}^{-1}$

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx \qquad f(x) \xrightarrow{\mathcal{F}} \hat{f}(\xi)$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x}\, d\xi$$

$x \to \xi \qquad \xi \to x$

$$2\pi f(\xi) = \int_{-\infty}^{\infty} \hat{f}(x) e^{ix\xi}\, dx$$

$\xi \to -\xi$

$$2\pi f(-\xi) = \int_{-\infty}^{\infty} \hat{f}(x) e^{-i\xi x}\, dx \qquad \hat{f}(x) \xrightarrow{\mathcal{F}} 2\pi f(-\xi)$$

Plancherel Theorem

Parseval's Identity

$$f(x) = \sum_n c_n e^{inx}, \quad g(x) = \sum_n d_n e^{inx}$$

$$\langle f, g \rangle = 2\pi \sum_n c_n \overline{d_n}, \quad \|f\|^2 = 2\pi \sum_n |c_n|^2$$

$$\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx \qquad \int_{-\pi}^{\pi} |f(x)|^2\,dx$$

Plancherel Theorem

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx, \quad \hat{g}(\xi) = \int_{-\infty}^{\infty} g(x) e^{-i\xi x} dx$$

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) \overline{g(x)} \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) \overline{\hat{g}(\xi)} \, d\xi$$

$$= \frac{1}{2\pi} \langle \hat{f}, \hat{g} \rangle$$

$$\|f\|^2 = \frac{1}{2\pi} \|\hat{f}\|^2$$

Further Properties of Fourier Transform (p. 214, p. 223)

x		ξ	
$f(x),\ g(x)$		$\hat{f}(\xi),\ \hat{g}(\xi)$	
$af(x)+bg(x)$	linearity	$a\hat{f}(\xi)+b\hat{g}(\xi)$	
$f(ax)$	(scaling property)	$\frac{1}{a}\hat{f}\left(\frac{\xi}{a}\right)$	
$f(x-c)$	(shift in x)	$e^{-ic\xi}\hat{f}(\xi)$	(modulated in ξ)
$e^{icx}f(x)$	(modulated in x)	$\hat{f}(\xi-c)$	(shift in ξ)
$f'(x)$	(differentiation in x)	$(i\xi)\hat{f}(\xi)$	(multiplication by ξ)
$xf(x)$		$i\hat{f}'(\xi)$	

Further Properties of Fourier Transform (p. 214 – p. 223)

x	ξ
$f(x),\ g(x)$	$\hat{f}(\xi),\ \hat{g}(\xi)$
$(f*g)(x)$ (convolution in x)	$\hat{f}(\xi)\,\hat{g}(\xi)$ (multiplication in ξ)
$f(x)\,g(x)$	$\frac{1}{2\pi}(\hat{f}*\hat{g})(\xi)$

Examples of Fourier Transform

(p. 214
p. 223)

(x)	$\xrightarrow{\mathcal{F}}$ (ξ)
$\chi_a(x) = \begin{cases} 1 & \lvert x\rvert < a \\ 0 & \lvert x\rvert > a \end{cases}$	$\frac{2}{\xi}\sin(a\xi)$
$\frac{1}{x}\sin(ax)$	$\pi\chi_a(\xi) = \begin{cases} \pi & \lvert\xi\rvert < a \\ 0 & \lvert\xi\rvert > a \end{cases}$
$e^{-a\lvert x\rvert}$	$\frac{2a}{(a^2+\xi^2)}$
$\frac{1}{(x^2+a^2)}$	$\frac{\pi}{a}e^{-a\lvert\xi\rvert}$
$e^{-ax^2/2}$	$\sqrt{\frac{2\pi}{a}}\,e^{-\xi^2/2a}$