MA 520: Boundary Value Problems of Differential Equations
Spring 2024, Final Exam
Instructor: Yip

- This test booklet has SIX QUESTIONS, totaling 140 points for the whole test. You have 120 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, closed note, with no electronic devices**.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.
- **As a rule of thumb, you should give explicit and useful answers.** No points will be given for just writing down some generically true statements. In other words, your answers should try to make use of all the information given in the question.
- **As a rule of thumb, you should only use those methods that have been covered in class.** If you use some other methods "for the sake of convenience", at our discretion, we might not give you any credit. You have the right to contest. In that event, **you will be asked to explain your answer using only what has been covered in class up to the point of time of this exam.**

Name: Answer Key (Major:)

Question	Score
1.(20 pts)	
2.(20 pts)	
3.(20 pts)	
4.(20 pts)	
5.(20 pts)	
6.(40 pts)	
Total (140 pts)	

Some useful formula

1. The solution of $x'(t) = ax(t) + b(t)$, $x(0) = x_0$, with a being a constant, is given by
$$x(t) = e^{at}x_0 + \int_0^t e^{a(t-s)}b(s)\,ds.$$

2. The eigenfunctions and eigenvalues of $L = \partial_{xx}$ on $(0,l)$, $L\varphi = \lambda\varphi$, are given by,

   (a) for Dirichlet boundary condition, $\varphi(0) = \varphi(l) = 0$:
$$\varphi_n(x) = \sin\left(\frac{n\pi x}{l}\right), \quad \lambda_n = -\frac{n^2\pi^2}{l^2}, \quad n = 1, 2, \ldots$$

   (b) for Neumann boundary condition, $\varphi_x(0) = \varphi_x(l) = 0$:
$$\varphi_n(x) = \cos\left(\frac{n\pi x}{l}\right), \quad \lambda_n = -\frac{n^2\pi^2}{l^2}, \quad n = 0, 1, 2, \ldots$$

   Note that,
$$\int_0^l \sin^2\left(\frac{n\pi x}{l}\right)dx = \int_0^l \cos^2\left(\frac{n\pi x}{l}\right)dx = \frac{l}{2} \text{ for } n \geq 1 \text{ while for } n = 0, \text{ we have } \int_0^l 1^2\,dx = l.$$

3. The Fourier transform and its inverse are given by:
$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx, \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi x}\,d\xi.$$

   The following are some formula and properties of Fourier transform:

1.	$f(x)$	$\hat{f}(\xi)$
2.	$f(x-c)$	$e^{-ic\xi}\hat{f}(\xi)$
3.	$e^{icx}f(x)$	$\hat{f}(\xi - c)$
4.	$f(ax)$	$a^{-1}\hat{f}(a^{-1}\xi)$
5.	$f'(x)$	$i\xi\hat{f}(\xi)$
6.	$xf(x)$	$i(\hat{f})'(\xi)$
7.	$(f * g)(x)$	$\hat{f}(\xi)\hat{g}(\xi)$
8.	$f(x)g(x)$	$(2\pi)^{-1}(\hat{f} * \hat{g})(\xi)$
9.	$e^{-ax^2/2}$	$\sqrt{2\pi/a}\,e^{-\xi^2/2a}$
10.	$(x^2+a^2)^{-1}$	$(\pi/a)e^{-a\lvert\xi\rvert}$
11.	$e^{-a\lvert x\rvert}$	$2a(\xi^2+a^2)^{-1}$
12.	$\chi_a(x) = \begin{cases} 1 & (\lvert x\rvert < a) \\ 0 & (\lvert x\rvert > a) \end{cases}$	$2\xi^{-1}\sin a\xi$
13.	$x^{-1}\sin ax$	$\pi\chi_a(\xi) = \begin{cases} \pi & (\lvert\xi\rvert < a) \\ 0 & (\lvert\xi\rvert > a) \end{cases}$

1. You are given the following information:

$$\begin{pmatrix} -5 & 2 \\ 2 & -8 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = -4\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -5 & 2 \\ 2 & -8 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = -9\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

(a) Solve the following system of differential equations,

$$\frac{dX}{dt}(t) = \begin{pmatrix} -5 & 2 \\ 2 & -8 \end{pmatrix} X(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(b) What can you say about $X(t)$ as $t \longrightarrow +\infty$? If $\lim_{t\to+\infty} X(t)$ exists, give an interpretation of this limit and demonstrate quantitatively your statement.

(a) Let $X(t) = c_1(t)\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2(t)\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = a\begin{pmatrix} 1 \\ 2 \end{pmatrix} + b\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$\perp$

$$a = \frac{\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \rangle}{\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \rangle} = \frac{3}{5}, \quad b = \frac{\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \rangle}{\langle \begin{pmatrix} -1 \\ 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \rangle} = \frac{1}{5}$$

Hence

$$\dot{X} = AX + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\dot{c}_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \dot{c}_2\begin{pmatrix} -1 \\ 2 \end{pmatrix} = -4c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} - 9c_2\begin{pmatrix} -1 \\ 2 \end{pmatrix} + \frac{3}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\dot{c}_1 = -4c_1 + \frac{3}{5}, \qquad \dot{c}_2 = -9c_2 + \frac{1}{5}$$

$$c_1(t) = e^{-4t}\cancel{c_1(0)}^{\,0} + \int_0^t e^{-4(t-s)}\frac{3}{5}\,ds, \qquad c_2(t) = e^{-9t}\cancel{c_2(0)}^{\,0} + \int_0^t e^{-9(t-s)}\frac{1}{5}\,ds$$

$$= e^{-4t}\left(\frac{3}{5}\left(\frac{1}{4}\right)\left(e^{4t}-1\right)\right) \qquad = e^{-9t}\frac{1}{5}\left(\frac{1}{9}\right)\left(e^{9t}-1\right)$$

$$= \frac{3}{20}\left(1-e^{-4t}\right) \qquad = \frac{1}{45}\left(1-e^{-9t}\right)$$

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$$\boxed{X(t) = \frac{3}{20}\left(1 - e^{-4t}\right)\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{1}{45}\left(1 - e^{-9t}\right)\begin{pmatrix} -1 \\ 2 \end{pmatrix}}$$

(b) $$\boxed{\lim_{t \to \infty} X(t) = \frac{3}{20}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{1}{45}\begin{pmatrix} -1 \\ 2 \end{pmatrix}}$$

$(\text{as } e^{-4t}, e^{-9t} \to 0)$

$$= \begin{pmatrix} \frac{3}{10} \\ \frac{3}{20} \end{pmatrix} + \begin{pmatrix} -\frac{1}{45} \\ \frac{2}{45} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{27}{90} - \frac{2}{90} \\ \frac{27}{180} + \frac{8}{180} \end{pmatrix} = \begin{pmatrix} \frac{25}{90} \\ \frac{35}{180} \end{pmatrix} = \begin{pmatrix} \frac{5}{18} \\ \frac{7}{36} \end{pmatrix}$$

steady state

$$\boxed{\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 & 2 \\ 2 & -8 \end{pmatrix}\begin{pmatrix} 5/18 \\ 7/36 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$$-\frac{25}{18} + \frac{7}{18} + 1 = 0$$

$$\frac{10}{18} - \frac{28}{18} + 1 = 0$$

Similar to [F, p120, #5]

2. Consider the annulus domain (expressed in polar coordinates): $\Omega = \{a \le r \le b;\ \ 0 \le \theta \le 2\pi\}$. The general form for a function u satisfying $\triangle u = 0$ in Ω is given by

$$u(r,\theta) = A_0 + B_0 \log r + \sum_{n=1}^{\infty} \left(A_n r^n + B_n r^{-n}\right) \cos n\theta + \left(C_n r^n + D_n r^{-n}\right) \sin n\theta. \quad (1)$$

Suppose u is required to satisfy the following boundary conditions,

$$\frac{\partial u}{\partial r}(a,\theta) = P - \sin 2\theta; \quad (2)$$
$$\frac{\partial u}{\partial r}(b,\theta) = Q + \cos 2\theta \quad (3)$$

where P and Q are some (given) constants.

(a) Find all the constants in (1) so that the boundary conditions (2) and (3) are satisfied. Are there any condition(s) for the constants P and Q such that u actually exists?

(b) When u exists, how many can you find?

(a) $$u_r = \frac{B_0}{r} + \sum_{n=1}^{\infty} \left(nA_n r^{n-1} - nB_n r^{-n-1}\right) \cos n\theta + \left(nC_n r^{n-1} - nD_n r^{-n-1}\right) \sin n\theta$$

$$r=a \Rightarrow \frac{B_0}{a} + \sum_{n=1}^{\infty} \left(nA_n a^{n-1} - nB_n a^{-n-1}\right) \cos n\theta + \left(nC_n a^{n-1} - nD_n a^{-n-1}\right) \sin n\theta = P - \sin 2\theta$$

$$r=b \Rightarrow \frac{B_0}{b} + \sum_{n=1}^{\infty} \left(nA_n b^{n-1} - nB_n b^{-n-1}\right) \cos n\theta \left(nC_n b^{n-1} - nD_n b^{-n-1}\right) \sin n\theta = Q + \cos 2\theta$$

$$\frac{B_0}{a} = P \quad \text{and} \quad 2C_2 a - 2D_2 a^{-3} = -1$$

and $$\frac{B_0}{b} = Q \quad \text{and} \quad 2A_2 b - 2B_2 b^{-3} = 1$$

and $$2C_2 b - 2D_2 b^{-3} = 0$$

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$$2A_2 a - 2B_2 a^{-3} = 0$$

C_2, D_2: $D_2 = C_2 b^4$, $\quad C_2 a - C_2 b^4 a^{-3} = -\frac{1}{2}$

$$C_2 = -\frac{1}{2} \frac{1}{a - b^4 a^{-3}}$$

$$= -\frac{1}{2} \frac{a^3}{a^4 - b^4}$$

$$\boxed{C_2 = \frac{1}{2} \frac{a^3}{b^4 - a^4} \qquad D_2 = \frac{1}{2} \frac{a^3 b^4}{b^4 - a^4}}$$

A_2, B_2: $B_2 = A_2 a^4$, $\quad A_2 b - A_2 a^4 b^{-3} = \frac{1}{2}$

$$\boxed{A_2 = \frac{1}{2} \frac{b^3}{b^4 - a^4} \qquad B_2 = \frac{1}{2} \frac{a^4 b^3}{b^4 - a^4}}$$

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All other A_n, B_n, C_n, D_n's are zero,

$$u(r,\theta) = A_0 + B_0 \log r$$

$$+ \left(\frac{1}{2}\frac{b^3}{b^4-a^4} r^2 + \frac{1}{2} \frac{a^4 b^3}{b^4-a^4} r^{-2} \right) \cos 2\theta$$

$$+ \left(\frac{1}{2}\frac{a^3}{b^4-a^4} r^2 + \frac{1}{2} \frac{a^3 b^4}{b^4-a^4} r^{-2} \right) \sin 2\theta$$

$\boxed{A_0 \text{ can be anything}, \quad B_0 = aP \text{ or } bQ}$

$\boxed{\text{Condition for } P, Q: \quad Pa = Qb}$

(b) when u exists (if $Pa = Qb$), there are infinitely many as A_0 is free (can be anything)

3. Consider the following heat equation on $(0, l)$:

$$\begin{aligned} u_t &= u_{xx} + h(x), \quad x \in (0,l), \ t > 0 \\ u_x(0,t) = 0, &\quad u_x(l,t) = 0, \quad t > 0 \\ u_(x,0) &= 0 \end{aligned}$$

where h is some given function defined on $(0, l)$.

(a) Solve for $u(x,t)$. Write down all the constants in your solution as explicitly as possible.

(b) Describe the behavior of $u(x,t)$ as $t \longrightarrow +\infty$.

(c) Under what condition(s) on h does $u(x,t)$ converge to a steady state as $t \longrightarrow +\infty$?

(a) Let $u(x,t) = \sum_{n=0}^{\infty} C_n(t) \cos\left(\frac{n\pi x}{l}\right)$

$$h(x) = \sum_{n=0}^{\infty} b_n \cos\left(\frac{n\pi x}{l}\right)$$

$$b_n = \frac{\int_0^l h(x) \cos\frac{n\pi x}{l}\, dx}{\int_0^l \cos^2\frac{n\pi x}{l}\, dx} = \begin{cases} \frac{1}{l}\int_0^l h(x)\, dx & n=0 \\ \frac{2}{l}\int_0^l h(x) \cos\frac{n\pi x}{l}\, dx & n \geq 1 \end{cases}$$

$$u_t = u_{xx} + h(x)$$

$$\Downarrow$$

$$\dot{C}_n = -\frac{n^2\pi^2}{l^2} C_n + b_n, \quad C_n(0) = 0$$

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$$C_0(t) = b_0 t$$

$$C_n(t) = \int_0^t e^{-\frac{n^2\pi^2}{\ell^2}(t-s)} b_n \, ds$$

$n \geq 1$

$$= e^{-\frac{n^2\pi^2}{\ell^2}t} b_n \frac{\ell^2}{n^2\pi^2}\left(e^{\frac{n^2\pi^2}{\ell^2}t} - 1\right)$$

$$= \frac{b_n \ell^2}{n^2\pi^2}\left(1 - e^{-\frac{n^2\pi^2}{\ell^2}t}\right)$$

$(n \neq 0)$

Hence

$$\boxed{u(x,t) = \underbrace{(b_0 t)}_{(n=0)} + \sum_{n=1}^{\infty} \frac{b_n \ell^2}{n^2\pi^2}\left(1 - e^{-\frac{n^2\pi^2}{\ell^2}t}\right)\cos nx}$$

(b) as $t \to +\infty$,

$$u(x,t) \sim b_0 t + \sum_{n=1}^{\infty} \frac{b_n \ell^2}{n^2\pi^2} \cos nx$$

linear growth in t

(c) If $b_0 = 0$, i.e. $\int_0^\ell h(x)\,dx = 0$, then

$$u(x,t) \to \sum_{n=1}^{\infty} \frac{b_n \ell^2}{n^2\pi^2} \cos nx = g(x), \text{ steady state}$$

$(0 = g_{xx} + h(x))$

[F, p. 184, #8]

4. Solve the initial value problem $u_t = k\Big[(1-x^2)u_x\Big]_x$, $u(x,0) = f(x)$ on $-1 < x < 1$. Here k is a positive constant. Find also the behavior of $u(x,t)$ as $t \longrightarrow +\infty$. Try to express your answer as explicitly or illustratively as possible but no need to dwell too much on "tedious" integrations.)

   (Note that the Legendre polynomials $\Big\{P_n(x)\Big\}_{n=0}^{\infty}$ satisfy

$$\Big[(1-x^2)P_n'(x)\Big]' + n(n+1)P_n(x) = 0 \quad \text{for } n \geq 0,$$

   and they form a complete orthogonal basis for $L^2(-1,1)$. If you find it useful, you can use: $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$ and so forth.)

$P_n(x)$ are eigenvectors of $L = \partial_x\big((1-x^2)\partial_x\big)$
with eigenvalue $-n(n+1)$

Let $\quad u(x,t) = \sum_{n=0}^{\infty} C_n(t) P_n(x)$

$$u_t = k\left[(1-x^2)u_x\right]_x$$

$$\Rightarrow \sum_{n=0}^{\infty} \dot{C}_n(t) P_n(x) = k \sum_{n=0}^{\infty} C_n(t) \big((1-x^2)P_n'(x)\big)'$$

$$= \sum_{n=0}^{\infty} -k\, n(n+1) C_n(t) P_n(x)$$

$$\Rightarrow \dot{C}_n(t) = -k\, n(n+1)\, C_n(t) \qquad n \geq 0$$

$$\Rightarrow C_n(t) = C_n(0)\, e^{-k(n)(n+1)t}$$

(Note: $C_0(t) = C_0(0)$.)

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$$C_n(0) = \frac{\langle f, P_n \rangle}{\langle P_n, P_n \rangle} = \frac{\int_{-1}^{1} f(x) P_n(x)\,dx}{\int_{-1}^{1} P_n^2(x)\,dx}, \quad n \geq 0$$

$$\boxed{u(x,t) = C_0(0) P_0(x) + \sum_{n=1}^{\infty} C_n(0) e^{-kn(n+1)t} P_n(x)}$$

as $t \to +\infty$

$$u(x,t) \longrightarrow C_0(0) P_0(x) = \left(\frac{1}{2}\int_{-1}^{1} f(x)\,dx\right)(1)$$

average of $f(x)$

Similar to [F, p235, #2]

5. Solve the following initial value problems

$$\begin{aligned} u_t(x,t) &= Du_{xx} + cu_x(x,t) + ku(x,t) + h(x,t), \quad x \in (-\infty, \infty) \\ u(x,0) &= f(x) \end{aligned}$$

where h and f are some given functions, and $D > 0, c, k$ are constants. Express your answer in "physical variables" x and t.

(Hint: use Fourier transform. You are given that the inverse Fourier transform of $e^{-D\xi^2 t}$ is given by $g(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$, or you can simply use g in your answer.)

g : $u(x,t) \longrightarrow \hat{u}(\xi,t)$

$$\hat{u}_t(\xi,t) = -D\xi^2 \hat{u} + ci\xi \hat{u} + k\hat{u} + \hat{h}(\xi,t)$$

$$= (-D\xi^2 + ci\xi + k)\hat{u}(\xi,t) + \hat{h}(\xi,t)$$

$$\hat{u}(\xi,t) = e^{(-D\xi^2 + ci\xi + k)t} \cancel{\hat{u}(\xi,0)} \; \hat{f}(\xi)$$

$$+ \int_0^t e^{(-D\xi^2 + ci\xi + k)(t-s)} \hat{h}(\xi,s)\, ds$$

$$= e^{-D\xi^2 t} e^{ci\xi t} e^{kt} \hat{f}(\xi)$$

$$+ \int_0^t e^{(-D\xi^2(t-s) + ci\xi(t-s) + k(t-s))} \hat{h}(\xi,s)\, ds$$

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$$e^{-D\xi^2 t} e^{Ci\xi t} e^{kt} \hat{f}(\xi) \xrightarrow{\mathcal{F}^{-1}}$$

$\mathcal{F}^{-1}$

$$\int_{-\infty}^{\infty} f(y) g(x-y,t)\,dy$$

$$\int_{-\infty}^{\infty} f(y) g(x+ct-y,t)\,dy$$

$$e^{kt}\int_{-\infty}^{\infty} f(y) g(x+ct-y,t)\,dy$$

$$\int_0^t e^{(-D\xi^2(t-s)+Ci\xi(t-s)+k(t-s))}\hat{h}(\xi,s)\,ds$$

$\mathcal{F}^{-1}$

$$\int_0^t \int h(y,s) g(x-y,t-s)\,dy\,ds$$

$$\int_0^t \int h(y,s) g(x+ct-y,t-s)\,dy\,ds$$

$$\int_0^t e^{k(t-s)} \int h(y,s) g(x+c(t-s)-y,t-s)\,dy\,ds$$

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Hence

$$\boxed{u(x,t) = e^{kt}\int_{-\infty}^{\infty} f(y)\, g(x+ct-y,t)\, dy + \int_0^t e^{k(t-s)}\int h(y,s)\, g(x+c(t-s)-y, t-s)\, dy\, ds}$$

6. Consider the three dimensional cylindrical domain: $\Omega = \Big\{ 0 \le x^2 + y^2 \le b,\ \ 0 \le z \le l \Big\}$ which is conveniently expressed in cylindrical coordinates as:

$$\Omega = \Big\{ 0 \le r \le b,\ \ 0 \le \theta < 2\pi,\ \ 0 \le z \le l \Big\}.$$

Consider the following boundary value problems:

(I)

[F, p.160, (5.49)]

$$\begin{aligned} \triangle u &= 0, \quad \text{in } \Omega \\ u(r,\theta,0) &= 0, \quad 0 \le r < b, \quad 0 \le \theta < 2\pi \\ u(r,\theta,l) &= 0, \quad 0 \le r < b, \quad 0 \le \theta < 2\pi \\ u(b,\theta,z) &= h(\theta,z), \quad 0 \le \theta < 2\pi, \quad 0 \le z \le l. \end{aligned}$$

(II)

[F, p.155, (5.46)
h ≡ 0]

$$\begin{aligned} \triangle u &= 0, \quad \text{in } \Omega \\ u(r,\theta,0) &= f(r,\theta), \quad 0 \le r < b, \quad 0 \le \theta < 2\pi \\ u(r,\theta,l) &= g(r,\theta), \quad 0 \le r < b, \quad 0 \le \theta < 2\pi \\ u(b,\theta,z) &= 0, \quad 0 \le \theta < 2\pi, \quad 0 \le z \le l. \end{aligned}$$

In each of the above problems, express the solution u using the form of separation of variables, $R(r)\Phi(\theta)Z(z)$. You need to write down all the possible R, Φ, and Z functions. In addition, identify/specify all the constants as explicitly or illustratively as possible. However, there is no need to dwell too much on "tedious" integrations.

Note the following.

(a) The Laplacian in cylindrical coordinates is given by

$$\triangle u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}$$

so that

$$\begin{aligned} &\triangle\Big[R(r)\Phi(\theta)Z(z)\Big] \\ &\quad = R''(r)\Phi(\theta)Z(z) + \frac{R'(r)}{r}\Phi(\theta)Z(z) + \frac{R(r)}{r^2}\Phi''(\theta)Z(z) + R(r)\Phi(\theta)Z''(z). \end{aligned}$$

Hence $\triangle\Big[R(r)\Phi(\theta)Z(z)\Big] = 0$ leads to

$$\frac{R''(r)}{R(r)} + \frac{1}{r}\frac{R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Phi''(\theta)}{\Phi(\theta)} + \frac{Z''(z)}{Z(z)} = 0.$$

(b) The eigenvalues of Laplace operator with Dirichlet boundary condition in any dimension is negative. In particular, if $\triangle\Big[R(r)\Phi(\theta)\Big] = \lambda\Big[R(r)\Phi(\theta)\Big]$ on the two dimensional disk $D = \{0 < r < b, 0 < \theta < 2\pi\}$ such that $R(b) = 0$ with $R(0^+)$ bounded, and Φ being 2π-periodic, then λ must be negative.

(c) The general solutions of

$$r^2R''(r) + rR'(r) + (\mu^2r^2 - \nu^2)R(r) = 0,$$

and

$$r^2R''(r) + rR'(r) - (\mu^2r^2 + \nu^2)R(r) = 0,$$

for $r > 0$ with $R(0^+)$ bounded are given by $R(r) = AJ_\nu(\mu r)$ and $R(r) = AI_\nu(\mu r)$ respectively, where $J_\nu(\cdot)$ is the Bessel function (of the first kind), $I_\nu(\cdot)$ is the *modified* Bessel function (of the first kind), and A is an arbitray constant.

Even though we have not talked about I_ν in class, for the purpose of this exam, you can imagine that I_ν is similar to J_ν, just a different type of function. One thing to note though, while J_ν can oscillate leading to infinitely many zeros, I_ν is always a positive function. To be specific, we have

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu} \quad \text{and} \quad I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}.$$

(I)

$$\underbrace{\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Phi''}{\Phi}}_{\text{function of } r,\ \theta} = -\underbrace{\frac{Z''}{Z}}_{\text{function of } z} = \alpha$$

$$Z'' = -\alpha Z, \qquad Z(0) = Z(\ell) = 0$$

① $Z = Z_n(z) = \sin\frac{n\pi z}{\ell}, \qquad \alpha = \alpha_n = \frac{n^2\pi^2}{\ell^2}$

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$$\frac{\Phi''}{\Phi} = -m^2, \qquad \Phi - 2\pi\text{-per in } \theta$$

$$\Phi = \Phi_m(\theta) = \sin m\theta, \quad m \geq 1$$

$$\cos m\theta, \quad m \geq 0$$

3

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{1}{r^2}m^2 = \frac{n^2\pi^2}{\ell^2}$$

$$\Rightarrow \quad r^2R'' + rR' - \Big(\underbrace{\frac{n^2\pi^2}{\ell^2}}_{\mu^2} r^2 + \underbrace{m^2}_{\nu^2}\Big) R = 0$$

Hence $R = R_{n,m}(r) = I_m\left(\frac{n\pi}{\ell} r\right)$

$$\boxed{u(r,\theta,z) = \sum_{n,m} A_{n,m}\, I_m\left(\frac{n\pi r}{\ell}\right) \Phi_m(\theta) \sin\frac{n\pi z}{\ell}}$$

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at $r = b \Rightarrow$

$$h(\theta, z) = \sum_{n,m} A_{n,m} I_m\left(\frac{n\pi b}{\ell}\right) \Phi_m(\theta) \sin\left(\frac{n\pi z}{\ell}\right)$$

$$A_{n,m} I_m\left(\frac{n\pi b}{\ell}\right) = \frac{\left\langle h, \Phi_m(\theta) \sin\left(\frac{n\pi z}{\ell}\right)\right\rangle}{\left\langle \Phi_m(\theta)\sin\frac{n\pi z}{\ell}, \Phi_m(\theta)\sin\frac{n\pi z}{\ell}\right\rangle}$$

$$= \frac{\int_0^\ell \int_0^{2\pi} h(\theta, z) \Phi_m(\theta) \sin\frac{n\pi z}{\ell}\, d\theta\, dz}{\int_0^\ell \int_0^{2\pi} \Phi_m^2(\theta) \sin^2\frac{n\pi z}{\ell}\, d\theta\, dz}$$

$$\boxed{A_{n,m} = \frac{1}{I_m\left(\frac{n\pi b}{\ell}\right)} \frac{\int_0^\ell \int_0^{2\pi} h(\theta, z) \Phi_m(\theta) \sin\frac{n\pi z}{\ell}\, d\theta\, dz}{\int_0^\ell \int_0^{2\pi} \Phi_m^2(\theta) \sin^2\frac{n\pi z}{\ell}\, d\theta\, dz}}$$

note: $I_m > 0, \neq 0$

II

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$$\underbrace{\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Phi''}{\Phi}}_{\text{function of } r,\ \theta} = \underbrace{-\frac{Z''}{Z}}_{\text{function of } z} = \alpha$$

$$\Longrightarrow \quad \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Phi''}{\Phi} = \alpha$$

$$\Longleftrightarrow \quad R''\Phi + \frac{1}{r}R'\Phi + \frac{1}{r^2}R\Phi'' = \alpha R\Phi$$

i.e. $\Delta(R\Phi) = \alpha(R\Phi)$

$$\Longrightarrow \quad \alpha = -\mu^2 < 0$$

(i) $\frac{\Phi''}{\Phi} = -m^2$, $\quad \Phi = \Phi_m(\theta) = \sin m\theta, \ \cos m\theta$

$m \geq 1 \qquad m \geq 0$

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(3)

$$\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}+\frac{1}{r^2}\frac{\Phi''}{\Phi}=\alpha$$

$$\Rightarrow \frac{R''}{R}+\frac{1}{r}\frac{R'}{R}-\frac{1}{r^2}m^2=-\mu^2$$

$$\Rightarrow r^2R''+rR'-(\mu^2r^2+m^2)R=0$$

$$\Rightarrow R=R_{m,k}(r)=J_m(\mu_k r)$$

$$R(b)=0 \Rightarrow J_m(\mu_k b)=0$$

Let $z_{m,k}$ be the k^{th} zero of J_m

i.e. $J_m(z_{m,k})=0$

Then $\mu_k b=z_{m,k}$, $\mu_k=\frac{z_{m,k}}{b}$

Hence $R_{m,k}(r)=J_m\left(\frac{z_{m,k}}{b}r\right)$

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3

$$\alpha = -\mu^2; \qquad \alpha_{m,k} = -\left(\frac{z_{m,k}}{b}\right)^2$$

$$\frac{Z''}{Z} = \alpha \implies Z'' = -\left(\frac{z_{m,k}}{b}\right)^2 Z$$

$$\implies Z(z) = A_{m,k} e^{\frac{z_{m,k}}{b} z} + B_{m,k} e^{-\frac{z_{m,k}}{b} z}$$

Hence

$$\boxed{u(r,\theta,z) = \sum_{m,k} \left(A_{m,k} e^{\frac{z_{m,k}}{b} z} + B_{m,k} e^{-\frac{z_{m,k}}{b} z}\right) \times J_m\left(\frac{z_{m,k}}{b} r\right) \Phi_m(\theta)}$$

$z = 0 \implies$

$$f(r,\theta) = \sum_{m,k} \left(A_{m,k} + B_{m,k}\right) J_m\left(\frac{z_{m,k}}{b} r\right) \Phi_m(\theta)$$

$z = l \implies$

$$g(r,\theta) = \sum_{m,k} \left(A_{m,k} e^{\frac{z_{m,k}}{b} l} + B_{m,k} e^{-\frac{z_{m,k}}{b} l}\right) J_m\left(\frac{z_{m,k}}{b} r\right) \Phi_m(\theta)$$

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$$\left(A_{m,k}+B_{m,k}\right) = \frac{\int_0^{2\pi}\int_0^b f(r,\theta)\, J_m\left(\frac{z_{m,k}}{b}r\right)\Phi_m(\theta)\, r\,dr\,d\theta}{\int_0^{2\pi}\int_0^b J_m^2\left(\frac{z_{k,m}}{b}r\right)\Phi_m^2(\theta)\, r\,dr\,d\theta}$$

$$A_{m,k}e^{\frac{z_{m,k}}{b}l} + B_{m,k}e^{-\frac{z_{m,k}}{b}l} = \frac{\int_0^{2\pi}\int_0^b g(r,\theta)\, J_m\left(\frac{z_{m,k}}{b}r\right)\Phi_m(\theta)\, r\,dr\,d\theta}{\int_0^{2\pi}\int_0^b J_m^2\left(\frac{z_{k,m}}{b}r\right)\Phi_m^2(\theta)\, r\,dr\,d\theta}$$

Solve for $A_{m,k}$, $B_{m,k}$.