MA 520: Boundary Value Problems of Differential Equations
Spring 2024, Midterm One
Instructor: Yip

- This test booklet has FOUR QUESTIONS, totaling 100 points for the whole test. You have 75 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, closed note, with no electronic devices**.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.
- **As a rule of thumb, you should give explicit and useful answers.** No points will be given for just writing down some generically true statements. In other words, your answers should try to make use of all the information given in the question.
- **As a rule of thumb, you should only use those methods that have been covered in class.** If you use some other methods "for the sake of convenience", at our discretion, we might not give you any credit. You have the right to contest. In that event, **you will be asked to explain your answer using only what has been covered in class up to the point of time of this exam.**

Name: Answer Key (Major:)

Question	Score
1.(25 pts)	
2.(25 pts)	
3.(25 pts)	
4.(25 pts)	
Total (100 pts)	

1. Consider the 2π-periodic function given by $f(x) = x^2$ on $-\pi < x < \pi$.

   (a) Find its Fourier series expansion.

   (b) Using the above or otherwise, compute the following series:

$$\begin{aligned}(A) &= \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \cdots \\ (B) &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \cdots \\ (C) &= \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} \cdots\end{aligned}$$

(a) x^2 – even function. Hence

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{3\pi}\pi^3 = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\, d\left(\frac{\sin nx}{n}\right)$$

$$= \frac{2}{\pi}\left[\cancel{\frac{x^2 \sin nx}{n}\Bigg|_0^{\pi}} - \frac{2}{n}\int_0^{\pi} x \sin nx\,dx \right]$$

$$= \frac{2}{\pi}\frac{2}{n}\left[\int_0^{\pi} x\, d\frac{\cos nx}{n}\right]$$

$$= \frac{4}{\pi n}\left[\frac{x\cos nx}{n_2}\Bigg|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\cos nx\,dx\right]$$

$$= \frac{4}{\pi n}\left[\frac{\pi\cos n\pi}{n} - \cancel{\frac{1}{n^2}\sin nx\Big|_0^\pi}\right]^0$$

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$$= \frac{4}{n^2}(-1)^n$$

Hence

$$\boxed{x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx,}$$

(b) $x=0 \Longrightarrow$

$$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n$$

$$= \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right]$$

(A) $$\boxed{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}\cdots = \frac{\pi^2}{12}}$$

$x=\pi \Longrightarrow$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n^2} \qquad \left(\cos n\pi = (-1)^n\right)$$

$$\frac{2\pi^2}{12} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

(B) $$\boxed{\text{i.e. } \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}}$$

(C) M1

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By Parseval Identity:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f|^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2$$

$$\frac{1}{\pi}\int_0^{\pi} x^4\,dx = \frac{\pi^4}{9} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\frac{1}{\pi}\frac{\pi^5}{5} = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

Hence $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{8}\left(\frac{1}{5}-\frac{1}{9}\right)\pi^4 = \frac{\pi^4}{90}$

(C) $$\boxed{\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots = \frac{\pi^4}{90}}$$

M2

From: $$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx$$

$\int dx$

$$\frac{x^3}{3} = \frac{\pi^2 x}{3} + \sum_{n=1}^{\infty}\frac{4}{n^3}(-1)^n\sin nx + \cancel{C}$$

$x=0 \Longrightarrow C=0$

$$\frac{x^3}{3} - \frac{\pi^2 x}{3} = \sum_{n=1}^{\infty} \frac{4}{n^3}(-1)^n \sin nx$$

$\int dx$

$$\frac{x^4}{12} - \frac{\pi^2 x^2}{6} = \sum_{n=1}^{\infty} \frac{4}{n^4}(-1)^{n+1} \cos nx + C$$

$\int_{-\pi}^{\pi} dx$

$$\frac{1}{60}\pi^5 \times 2 - \frac{\pi^2}{18}\pi^3 \times 2 = C\,2\pi$$

$$\left(\frac{1}{30} - \frac{1}{9}\right)\frac{\pi^5}{2\pi} = C$$

$$C = \frac{-21}{540}\pi^4 = -\frac{7\pi^4}{180}$$

Hence $$\frac{x^4}{12} - \frac{\pi^2 x^2}{6} = -\frac{7\pi^4}{180} + \sum_{n=1}^{\infty} \frac{4}{n^4}(-1)^{n+1} \cos nx$$

$\cos n\pi = (-1)^n$

$x = \pi \Rightarrow$

$$\left(\frac{1}{12} - \frac{1}{6} + \frac{7}{180}\right)\pi^4 = 4(-1)\sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\left(-\frac{15 - 30 + 7}{4(180)}\right)\pi^4 = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$= \frac{\pi^4}{90}$$

2. Let D be the unit disk $\{x^2+y^2 \le 1\}$ in $\mathbb{R}^2$. Let

$$L^2(D) = \left\{ f : \iint_D |f(x,y)|^2 \, dxdy < \infty \right\}, \quad \langle f, g \rangle = \iint_D f(x,y)\overline{g(x,y)} \, dxdy.$$

Introduce $\mathcal{L} = \{f_n(x,y) = (x+yi)^n\}_{n=0}^{\infty}$.

(a) Show that $\mathcal{L}$ is an orthogonal list of functions. Find also $\|f_n\|$.

(Hint: use polar coordinates $x+iy = re^{i\theta}$ and the formula $dxdy = rdrd\theta$.)

(b) Find the projections of the functions $f(x,y) = x$ and $g(x,y) = y$ onto the space spanned by $\mathcal{L}$, i.e., find $\mathrm{Proj}_{\mathcal{L}} x$ and $\mathrm{Proj}_{\mathcal{L}} y$.

(Hint: write x and y using polar coordinates.)

(c) Is $\mathcal{L}$ complete in $L^2(D)$?

(a) $$\iint_D f_n \overline{f_m} \, dxdy = \int_0^1 \int_0^{2\pi} (r^n e^{in\theta}) \overline{(r^m e^{im\theta})} \, r \, d\theta \, dr$$

$$= \int_0^1 r^{n+m+1} \underbrace{\left(\int_0^{2\pi} e^{i(n-m)\theta} \, d\theta \right)}_{n \neq m \;=\; \left. \frac{e^{i(n-m)\theta}}{i(n-m)} \right|_0^{2\pi} \;=\; 0} dr$$

Hence $f_n \perp f_m$ in $L^2(D)$ for $n \neq m$

when $n=m$,

$$\langle f_n, f_n \rangle = \int_0^1 r^{2n+1} \int_0^{2\pi} d\theta \, dr = \frac{2\pi}{2n+2} = \frac{\pi}{n+1}$$

i.e. $\boxed{\|f_n\|^2 = \frac{\pi}{n+1}, \text{ or } \|f_n\| = \sqrt{\frac{\pi}{n+1}}}$

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(b) $\text{Proj}_{\mathcal{L}} x = \sum_{n=0}^{\infty} \frac{\langle x, f_n \rangle}{\|f_n\|^2} f_n$

$$\langle x, f_n \rangle = \int_D x \, \overline{(x+yi)^n} \, dx\,dy$$

$(x = r\cos\theta)$

$$= \int_0^1 \int_0^{2\pi} (r\cos\theta)\, r^n e^{-in\theta}\, r\, d\theta\, dr$$

$$= \int_0^1 r^{n+2} \int_0^{2\pi} \cos\theta \, e^{-in} \, d\theta \, dr$$

$\left(\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}\right)$

$$= \left(\int_0^1 r^{n+2} dr\right)\left(\int_0^{2\pi} \frac{e^{i\theta} + e^{-i\theta}}{2} e^{-in\theta} d\theta\right)$$

$$= \left(\frac{1}{n+3}\right) \frac{1}{2} \int_0^{2\pi} \left(e^{(1-n)i\theta} + \cancel{e^{-(1+n)i\theta}}\right) d\theta$$

$= 0$ for $n = 0, 1, 2, \cdots$

$= 0$ for $n \neq 1$, $(n = 0, 2, 3, \cdots)$

when $n = 1$, $\langle x, f_1 \rangle = \left(\frac{1}{1+3}\right)\left(\frac{1}{2}\right) \int_0^{2\pi} d\theta = \frac{\pi}{4}$

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Hence $$\boxed{\widehat{\mathrm{Proj}}_{\mathcal{L}}\, x = \frac{\langle x, f_1\rangle}{\|f_1\|^2} f_1 = \frac{\frac{\pi}{4}}{\frac{\pi}{2}} f_1 = \frac{1}{2} f_1(x,y) = \frac{x+yi}{2}}$$

Similarly, $$\widehat{\mathrm{Proj}}_{\mathcal{L}}\, y = \sum_{n=0}^{\infty} \frac{\langle y, f_n\rangle}{\|f_n\|^2} f_n$$

$$\langle y, f_n\rangle = \iint_D y\, \overline{f_n}\, dx\, dy$$

$$= \int_0^1 \int_0^{2\pi} (r\sin\theta)\, r^n e^{-in\theta}\, r\, d\theta\, dr$$

$$= \left(\int_0^1 r^{n+2}\, dr\right) \int_0^{2\pi} (\sin\theta)\, e^{-in\theta}\, d\theta$$

$$\left(\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}\right)$$

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$$= \left(\frac{1}{n+3}\right)\frac{1}{2i}\int_0^{2\pi}\left(e^{i\theta}-e^{-i\theta}\right)e^{-in\theta}\,d\theta$$

$$= \frac{1}{(n+3)}\left(\frac{1}{2i}\right)\int_0^{2\pi}\left(\underbrace{e^{(1-n)i\theta}}_{=0 \text{ for } n\neq 1} - \underbrace{e^{-(1+n)i\theta}}_{=0 \text{ for } n=0,1,\dots}\right)d\theta$$

For $n=1$, $\langle y, f_1\rangle = \frac{1}{4}\left(\frac{1}{2i}\right)\int_0^{2\pi} d\theta = \frac{\pi}{4i}$

Hence

$$\boxed{\begin{aligned} \text{Proj}_{\mathcal{L}}\, y &= \frac{\langle y, f_1\rangle}{\|f_1\|^2} f_1 \qquad \frac{1}{2i} f_1(x,y) \\ &= \frac{\frac{\pi}{4i}}{\frac{\pi}{2}}(x+yi) = \frac{1}{2i}(x+yi) \end{aligned}}$$

(Note: $\text{Proj}_{\mathcal{L}}(x+yi) = \text{Proj}_{\mathcal{L}}\, x + i\,\text{Proj}_{\mathcal{L}}\, y$

$$= \frac{1}{2}(x+yi) + i\,\frac{1}{2i}(x+yi) = x+yi)$$

(c) $\mathcal{L}$ is not complete on $L^2(D)$. For otherwise, we would have $\text{Proj}_{\mathcal{L}}\, x = x$ and $\text{Proj}_{\mathcal{L}}\, y = y$

3. Consider the space of (real valued) functions defined on $(0, 1)$ with the following inner product:

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx.$$

$= 1$

You are given $P_1(x) = 1$, $P_2(x) = x$, and $P_3(x) = x^2$. Let $Q_1 = P_1$. (For simplicity, I have omitted the x's.)

(a) Find $Q_2 = P_2 - \text{Proj}_{\{Q_1\}}P_2$.

(b) Find $Q_3 = P_3 - \text{Proj}_{\{Q_1,Q_2\}}P_3$.

(c) Show that $\{Q_1, Q_2, Q_3\}$ forms an orthogonal set.

(a) $$Q_2 = P_2 - \text{Proj}_{Q_1} P_2 = x - \frac{\langle P_2, Q_1\rangle}{\langle Q_1, Q_1\rangle} Q_1$$

$$= x - \frac{\int_0^1 x \cdot 1\,dx}{\int_0^1 1 \cdot 1\,dx} 1 = x - \frac{\frac{1}{2}}{1}(1) = x - \frac{1}{2}$$

(b) $$Q_3 = P_3 - \text{Proj}_{\{Q_1, Q_2\}} P_3$$

$$= x^2 - \frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle} 1 - \frac{\langle x^2, x - \frac{1}{2}\rangle}{\langle x - \frac{1}{2}, x - \frac{1}{2}\rangle}\left(x - \frac{1}{2}\right)$$

$$= x^2 - \frac{\int_0^1 x^2\,dx}{\int_0^1 1\,dx} 1 - \frac{\int_0^1 x^2\left(x - \frac{1}{2}\right)dx}{\int_0^1 \left(x - \frac{1}{2}\right)^2 dx}\left(x - \frac{1}{2}\right)$$

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$$= x^2 - 1/3 - \frac{\int_0^1 (x^3 - \frac{x^2}{2})\,dx}{\int_0^1 (x^2 - x + \frac{1}{4})\,dx} (x - \frac{1}{2})$$

$$= x^2 - \frac{1}{3} - \left(\frac{\frac{1}{4} - \frac{1}{6}}{\frac{1}{3} - \frac{1}{2} + \frac{1}{4}}\right)(x - \frac{1}{2})$$

$$= x^2 - \frac{1}{3} - \frac{\frac{1}{\cancel{12}}}{\frac{4-6+3}{\cancel{12}}} (x - \frac{1}{2})$$

$$= x^2 - \frac{1}{3} - x + \frac{1}{2} = x^2 - x + \frac{1}{6}$$

(c) $Q_2 = P_2 - \mathrm{Proj}_{Q_1} P_2 \perp Q_1 \implies \underline{Q_1 \perp Q_2}$

M1

$Q_3 = P_3 - \mathrm{Proj}_{\{Q_1, Q_2\}} P_3 \perp \{Q_1, Q_2\}$

$\implies \underline{Q_3 \perp Q_1},$

$\underline{Q_3 \perp Q_2}$

Hence $\{Q_1, Q_2, Q_3\}$ is orthogonal

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M2

$$\langle Q_1, Q_2 \rangle = \int_0^1 1\left(x - \frac{1}{2}\right) dx = \frac{1}{2} - \frac{1}{2} = 0$$

$$\langle Q_1, Q_3 \rangle = \int_0^1 1\left(x^2 - x + \frac{1}{6}\right) dx = \frac{1}{3} - \frac{1}{2} + \frac{1}{6} = 0$$

$$\langle Q_2, Q_3 \rangle = \int_0^1 \left(x - \frac{1}{2}\right)\left(x^2 - x + \frac{1}{6}\right) dx$$

$$= \int_0^1 \left(x^3 - x^2 + \frac{x}{6} - \frac{x^2}{2} + \frac{x}{2} - \frac{1}{12}\right) dx$$

$$= \frac{1}{4} - \frac{1}{3} + \cancel{\frac{1}{12}} - \frac{1}{6} + \frac{1}{4} - \cancel{\frac{1}{12}}$$

$$= \frac{1}{2} - \frac{1}{3} - \frac{1}{6} = 0$$

4. You are given the following information:

$$\begin{pmatrix} -5 & 2 \\ 2 & -8 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = -4 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -5 & 2 \\ 2 & -8 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = -9 \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

(a) Solve the following system

$$\frac{d^2X}{dt^2}(t) = \begin{pmatrix} -5 & 2 \\ 2 & -8 \end{pmatrix} X(t), \quad X(0) = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad \frac{dX}{dt}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

What can you say about the solution as $t \longrightarrow +\infty$?

(b) Consider the following system

$$\frac{d^2X}{dt^2}(t) + \frac{dX}{dt}(t) = \begin{pmatrix} -5 & 2 \\ 2 & -8 \end{pmatrix} X(t),$$

(with some unknown/unspecified initial conditions).

Find the general solution. You should write down the solution as precise as possible. (Of course not all constants can be identified/found as the initial conditions are not given.)

What can you say about the solution as $t \longrightarrow +\infty$?

(a) $X(t) = a_1(t)\, V_1 + a_2(t)\, V_2$

$\ddot{X} = AX$

$\ddot{a}_1 V_1 + \ddot{a}_2 V_2 = A(a_1 V_1 + a_2 V_2) = a_1 \lambda_1 V_1 + a_2 \lambda_2 V_2$

$\ddot{a}_1 = -4 a_1$ and $\ddot{a}_2 = -9 a_2$

Hence $a_1(t) = (a \cos 2t + b \sin 2t)$

$a_2(t) = (c \cos 3t + d \sin 3t)$

$$X(t) = (a\cos 2t + b\sin 2t)V_1 + (c\cos 3t + d\sin 3t)V_2$$

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$$= (a\cos 2t + b\sin 2t)\begin{pmatrix}2\\1\end{pmatrix} + (c\cos 3t + d\sin 3t)\begin{pmatrix}-1\\2\end{pmatrix}$$

$$X(0) = a\begin{pmatrix}2\\1\end{pmatrix} + c\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}1\\4\end{pmatrix}$$

$$a = \frac{\left\langle \begin{pmatrix}1\\4\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix} \right\rangle} = \frac{6}{5}, \qquad c = \frac{\left\langle \begin{pmatrix}1\\4\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix}-1\\2\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix} \right\rangle} = \frac{7}{5}$$

$$\dot{X}(t) = (-2a\sin 2t + 2b\cos 2t)\begin{pmatrix}2\\1\end{pmatrix} + (-3c\sin 3t + 3d\cos 3t)\begin{pmatrix}-1\\2\end{pmatrix}$$

$$\dot{X}(0) = (2b)\begin{pmatrix}2\\1\end{pmatrix} + (3d)\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}$$

$$2b = \frac{\left\langle \begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix} \right\rangle} = \frac{3}{5} \Longrightarrow b = \frac{3}{10}$$

$$3d = \frac{\left\langle \begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix}-1\\2\end{pmatrix}, \begin{pmatrix}-1\\2\end{pmatrix} \right\rangle} = -\frac{4}{5} \Longrightarrow d = -\frac{4}{15}$$

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$$\boxed{X(t) = \left(\frac{6}{5}\cos 2t + \frac{3}{10}\sin 2t\right)\begin{pmatrix}2\\1\end{pmatrix} + \left(\frac{7}{5}\cos 3t - \frac{4}{15}\sin 3t\right)\begin{pmatrix}-1\\2\end{pmatrix}}$$

as $t \to +\infty$, $X(t)$ keeps oscillation.

(b) $\ddot{X} + \dot{X} = AX$

Let $X(t) = a_1(t) V_1 + a_2(t) V_2$

Then $(\ddot{a}_1 V_1 + \ddot{a}_2 V_2) + (\dot{a}_1 V_1 + \dot{a}_2 V_2) = a_1 \lambda_1 V_1 + a_2 \lambda_2 V_2$

$\ddot{a}_1 + \dot{a}_1 = -4a_1$ and $\ddot{a}_2 + \dot{a}_2 = -9a_2$

$r^2 + r + 4 = 0$

$r = \dfrac{-1 \pm \sqrt{1-16}}{2}$

$= -\dfrac{1}{2} \pm \dfrac{\sqrt{15}}{2} i$

$r^2 + r + 9 = 0$

$r = \dfrac{-1 \pm \sqrt{1-36}}{2}$

$= -\dfrac{1}{2} \pm \dfrac{\sqrt{35}}{2} i$

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Hence
$$a_1(t) = e^{-\frac{1}{2}t}\left(a\cos\frac{\sqrt{15}}{2}t + b\sin\frac{\sqrt{15}}{2}t\right)$$
$$a_2(t) = e^{-\frac{1}{2}t}\left(c\cos\frac{\sqrt{35}}{2}t + d\sin\frac{\sqrt{35}}{2}t\right)$$

$$\boxed{\begin{aligned} X(t) = {} & e^{-\frac{1}{2}t}\left(a\cos\frac{\sqrt{15}}{2}t + b\sin\frac{\sqrt{15}}{2}t\right)\begin{pmatrix}2\\1\end{pmatrix} \\ & e^{-\frac{1}{2}t}\left(c\cos\frac{\sqrt{35}}{2}t + d\sin\frac{\sqrt{35}}{2}t\right)\begin{pmatrix}-1\\2\end{pmatrix} \end{aligned}}$$

(a, b, c, d cannot be determined as initial conditions are not specified.)

Note that as $t \to +\infty$, $X(t) \to 0$