MA 520: Boundary Value Problems of Differential Equations
Spring 2024, Midterm Two
Instructor: Yip

- This test booklet has FOUR QUESTIONS, totaling 100 points for the whole test. You have 75 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, closed note, with no electronic devices**.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.
- **As a rule of thumb, you should give explicit and useful answers.** No points will be given for just writing down some generically true statements. In other words, your answers should try to make use of all the information given in the question.
- **As a rule of thumb, you should only use those methods that have been covered in class.** If you use some other methods "for the sake of convenience", at our discretion, we might not give you any credit. You have the right to contest. In that event, **you will be asked to explain your answer using only what has been covered in class up to the point of time of this exam.**

Name: Answer Key (Major:)

Question	Score
1.(25 pts)	
2.(25 pts)	
3.(25 pts)	
4.(25 pts)	
Total (100 pts)	

(cf. [F, p. 120 #5b])

1. Consider the annulus domain (expressed in polar coordinates): $D = \{a \le r \le b;\ \ 0 \le \theta \le 2\pi\}$. The general form for a function u satisfying $\triangle u = 0$ in D is given by

$$u(r,\theta) = A_0 + B_0 \log r + \sum_{n=1}^{\infty} \left(A_n r^n + B_n r^{-n}\right)\cos n\theta + \left(C_n r^n + D_n r^{-n}\right)\sin n\theta.$$

Find all the constants if u satisfies the following boundary values:

$$\begin{aligned} u(a,\theta) &= 2 + \cos\theta - \sin 2\theta; \\ u(b,\theta) &= \cos 2\theta. \end{aligned}$$

$r = a \Rightarrow$

$$2 + \cos\theta - \sin 2\theta = A_0 + B_0 \log a + \sum_{n=1}^{\infty} \left(A_n a^n + B_n a^{-n}\right)\cos n\theta + \left(C_n a^n + D_n a^{-n}\right)\sin n\theta$$

$r = b \Rightarrow$

$$\cos 2\theta = A_0 + B_0 \log b + \sum_{n=1}^{\infty} \left(A_n b^n + B_n b^{-n}\right)\cos n\theta + \left(C_n b^n + D_n b^{-n}\right)\sin n\theta$$

I $\begin{cases} A_0 + B_0 \log a = 2 \\ A_0 + B_0 \log b = 0, \end{cases}$

$\Rightarrow B_0(\log a - \log b) = 2$

$\Rightarrow$ $\boxed{B_0 = \dfrac{2}{\log a - \log b}, \qquad A_0 = -\dfrac{2\log b}{\log a - \log b}}$

II $\begin{cases} A_1 a + B_1 a^{-1} = 1 \\ A_1 b + B_1 b^{-1} = 0, \end{cases}$

$\Rightarrow B_1(-ab^{-2} + a^{-1}) = 1$ $\Rightarrow$ $\boxed{B_1 = \dfrac{1}{-\frac{a}{b^2} + \frac{1}{a}} = \dfrac{ab^2}{b^2 - a^2} \qquad A_1 = -\dfrac{a}{b^2 - a^2}}$

You can use this blank page.

III $\begin{cases} A_2 a^2 + B_2 a^{-2} = 0 \\ A_2 b^2 + B_2 b^{-2} = 1 \end{cases} \Longrightarrow B_2\left(-b^2 a^{-4} + b^{-2}\right) = 1$

$$\boxed{\begin{aligned} B_2 &= \frac{1}{-\frac{b^2}{a^4} + \frac{1}{b^2}} = \frac{a^4 b^2}{a^4 - b^4} \\ A_2 &= -\frac{b^2}{a^4 - b^4} \end{aligned}}$$

IV $\begin{cases} C_2 a^2 + D_2 a^{-2} = -1 \\ C_2 b^2 + D_2 b^{-2} = 0, \end{cases} \Longrightarrow D_2\left(-a^2 b^{-4} + a^{-2}\right) = -1$

$$\boxed{\begin{aligned} D_2 &= -\frac{1}{-\frac{a^2}{b^4} + \frac{1}{a^2}} \\ &= -\frac{a^2 b^4}{b^4 - a^4} \\ C_2 &= \frac{a^2}{b^4 - a^4} \end{aligned}}$$

all other coeff's are zero.

[F, p. 114, #5, 6]

2. The eigenfunctions φ_n and eigenvalues λ_n for the operator $L = \partial_{xx}$ on the interval $[0, l]$ with Dirichlet boundary condition $\varphi_n(0) = \varphi_n(l) = 0$ are given by

$$\varphi_n(x) = \sin\left(\frac{n\pi x}{l}\right), \quad \lambda_n = -\frac{n^2\pi^2}{l^2}, \quad n = 1, 2, 3, \ldots, \quad (L\varphi_n = \lambda_n \varphi_n)$$

Consider the following partial differential equations defined on the interval $[0, l]$ with the above mentioned Dirichlet boundary condition.

(a) $u_{tt} = c^2 u_{xx} - ku$;

(b) $u_{tt} = c^2 u_{xx} - \mu u_t$.

The k and μ are positive constants with $\mu < \frac{2c\pi}{l}$.

Write the solution $u(x,t)$ as:

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\varphi_n(x).$$

For each case, find the *general solution form* for the $a_n(t)$'s. You *do not* need to match the $a_n(0)$'s and $\dot{a}_n(0)$'s with the initial data of u as they are not anyway specified.

(a) $$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\varphi_n(t)$$

$$\Downarrow$$

$$u_{tt} = c^2 u_{xx} - k u$$

$$\Rightarrow \sum_{n=1}^{\infty} \ddot{a}_n \varphi_n = \sum_{n=1}^{\infty}\left(c^2 a_n \lambda_n \varphi_n - k a_n \varphi_n\right)$$

$$\left((\varphi_n)_{xx} = \lambda_n \varphi_n\right)$$

$$\Rightarrow \ddot{a}_n = c^2\lambda_n a_n - k a_n = (c^2\lambda_n - k)a_n$$

$$= -\left(\frac{c^2\pi^2 n^2}{l^2} + k\right)a_n$$

Characteristic eqn: $r^2+\frac{c^2\pi^2n^2}{\ell^2}+k=0 \Rightarrow r=\pm\sqrt{\frac{c^2\pi^2n^2}{\ell^2}+k}\,i$

You can use this blank page.

$$\boxed{a_n(t) = A_n \cos\left(\sqrt{\frac{c^2\pi^2n^2}{\ell^2}+k}\,t\right)+B_n \sin\left(\sqrt{\frac{c^2\pi^2n^2}{\ell^2}+k}\,t\right)}$$

(b) $$u_{tt} = c^2u_{xx} - k\,u_t$$

$$\sum_n \ddot{a}_n\varphi_n = \sum_n c^2\lambda_n a_n\varphi_n - \sum_k k\,\dot{a}_n\varphi_n$$

$$\Rightarrow \quad \ddot{a}_n + k\,\dot{a}_n - c^2\lambda_n a_n = 0$$

Characteristic equation:

$$r^2+\mu r+\frac{c^2\pi^2n^2}{\ell^2}=0 \Rightarrow r=\frac{-\mu\pm\sqrt{\mu^2-\frac{4c^2\pi^2n^2}{\ell^2}}}{2}$$

<0 for all n as $\mu<\frac{2c\pi}{\ell}$

$$\Rightarrow$$

$$\boxed{a_n(t)= e^{-\frac{\mu t}{2}}\left[A_n\cos\left(\frac{1}{2}\sqrt{\frac{4c^2n^2\pi^2}{\ell^2}-\mu^2}\,t\right)+B_n\sin\left(\frac{1}{2}\sqrt{\frac{4c^2n^2\pi^2}{\ell^2}-\mu^2}\,t\right)\right]}$$

[F, p. 108 #8]

3. Consider the following initial boundary value problem:

$$\begin{aligned} u_t &= ku_{xx}, \quad 0<x<l; \\ u_x(0,t)=0, \quad & u_x(l,t)+bu(l,t)=0, \quad (b>0) \\ u(x,0) &= u_0(x) \end{aligned}$$

(a) Determine all the eigenvalues and eigenfunctions for the operator $L=\partial_{xx}$ with corresponding boundary conditions, i.e. find all λ and $\varphi \not\equiv 0$ such that $L\varphi=\lambda\varphi$.

i. You can *assume that all the λ are negative.*

ii. You might not be able to find the λ's analytically, but you should be able to describe how to find them and determine how many λ's there are. It might be best to give a grahical illustration.

(b) Solve for $u(x,t)$ using the eigenvalues and eigenfunctions you have just found.

(c) Find all the constants explicitly (in terms of the λ's) in the expression for $u(x,t)$ when $u_0(x)=100$.

(a) $\varphi_{xx}=\lambda\varphi, \quad \varphi_x(0)=0, \quad \varphi_x(l)+b\varphi(l)=0$

$\lambda<0 \Rightarrow \lambda=-\mu^2 \ (\neq 0)$

$\varphi_{xx}=-\mu^2\varphi \Rightarrow \varphi=A\cos\mu x+B\sin\mu x$

$\varphi_x(x)=-\mu A\sin\mu x+\mu B\cos\mu x$

$\varphi_x(0)=0 \Rightarrow B=0$, i.e. $\varphi_x(x)=-\mu A\sin\mu$

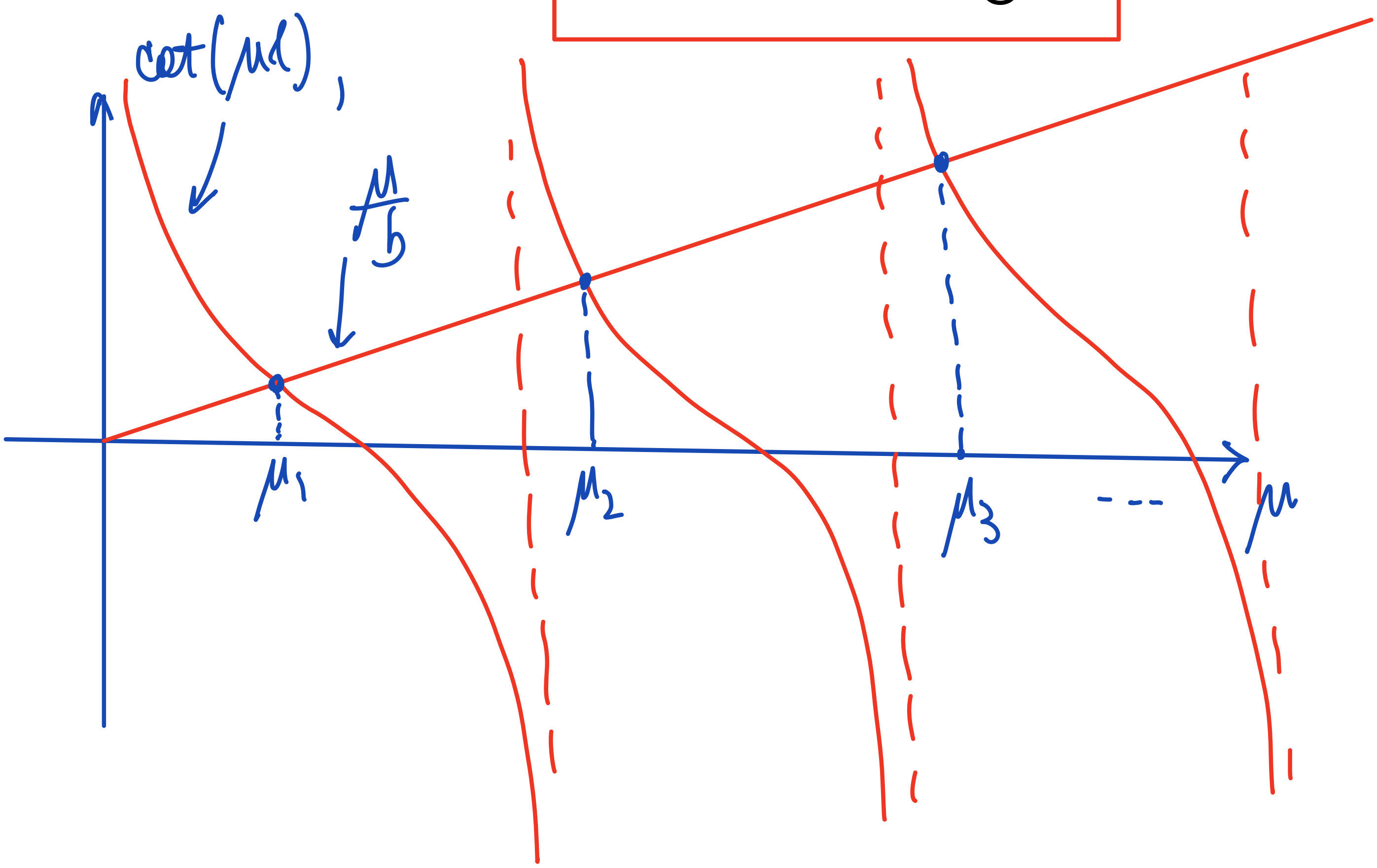
$\varphi(x)=A\cos\mu x, \quad \varphi_x(x)=-\mu A\sin\mu x$

$\varphi_x(l)+b\varphi(l)=0 \Longrightarrow A\underbrace{\left(-\mu\sin\mu l+b\cos\mu l\right)}_{=0}=0$

$\|$

You can use this blank page.

$$\boxed{\cot(\mu l) = \frac{\mu}{b}}$$



$$\boxed{\begin{array}{l} \mu_n\text{'s are solutions of } \cot(\mu l) = \frac{\mu}{b} \\ \lambda_n = -\mu_n^2 \leftarrow \text{eigenvalues} \\ \varphi_n(x) = \cos(\mu_n x) \end{array}}$$

(b) $u_t = k u_{xx}$

Let $u(x,t) = \sum_{n=1}^{\infty} a_n(t)\varphi_n(x)$

$$\Rightarrow \sum_{n=1}^{\infty} \dot{a}_n(t)\varphi_n(x) = \sum_{n=1}^{\infty} -k\mu_n^2 a_n(t)\,\varphi_n(x)$$

$$\dot{a}_n(t) = -k\mu_n^2 a_n(t) \Longrightarrow \boxed{a_n(t) = a_n(0)\, e^{-k\mu_n^2 t}}$$

You can use this blank page.

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n(0)\, e^{-k\mu_n^2 t} \cos(\mu_n x)}$$

$\lambda_n = -\mu_n^2$

$\mu_n = \sqrt{-\lambda_n}$

$t=0$

$$u_0(x) = \sum_{n=1}^{\infty} a_n(0) \cos(\mu_n x)$$

orthogonal

$$a_n(0) = \frac{\langle u_0, \cos\mu_n x\rangle}{\langle \cos\mu_n x, \cos\mu_n x\rangle} = \frac{\int_0^l u_0(x)\cos\mu_n x\, dx}{\int_0^l \cos^2(\mu_n x)\, dx}$$

(c) $u_0(x) \equiv 100$

$$a_n(0) = \frac{100 \int_0^l \cos\mu_n x\, dx}{\int_0^l \cos^2(\mu_n x)\, dx} = \frac{100 \frac{1}{\mu_n} \sin\mu_n x \Big|_0^l}{\frac{1}{2}\int_0^l (\cos(2\mu_n x)+1)\, dx}$$

$$= \frac{\left(\frac{200}{\mu_n}\right)\sin\mu_n l}{\frac{1}{2\mu_n}\sin(2\mu_n l) + l} = \boxed{\frac{400 \sin(\mu_n l)}{\sin(2\mu_n l) + 2\mu_n l}}$$

(cf. Past Exam B #5, [F (3.5) #1, #2 #12])

4. Consider the following Sturm-Liouville operator L on $r > 0$ and its (*weighted*) eigenvalue problem

$$LR = (rR'(r))' - \frac{\nu^2}{r}R(r), \quad LR(r) = \lambda r R(r).$$

(Note that the above is equivalent to the *Bessel equation of type* ν upon identifying $\lambda = -\mu^2$: $r^2R''(r) + rR'(r) + (\mu^2r^2 - \nu^2)R(r) = 0$.)

Restrict L to the interval $r \in [a, b]$ (with $a, b > 0$) and consider the following boundary conditions:

$$R'(a) = \alpha R(a), \;\; R'(b) = \beta R(b).$$

which are to be imposed for all the functions appearing in the following.

(a) Show that L is self-adjoint, i.e. $\int_a^b (LP)(r)Q(r)\,dr = \int_a^b P(r)(LQ)(r)\,dr$.

(b) Suppose P and Q satisfy $LP = \lambda_1 rP$, $LQ = \lambda_2 rQ$ with $\lambda_1 \neq \lambda_2$. Show that P is orthogonal to Q in the *weighted-L^2 space* $L^2_r(a,b)$, i.e. $\int_a^b P(r)Q(r)r\,dr = 0.$.

(c) Suppose $\beta < 0 < \alpha$. Then any eigenvalue λ must be *negative*.

(a)

$$\boxed{\int_a^b (LP)Q\,dr} = \int_a^b \left((rP')' - \frac{\nu^2}{r}P\right)Q\,dr$$

$$= rP'Q\Big|_a^b - \int_a^b rP'Q'\,dr - \int_a^b \frac{\nu^2}{r}PQ\,dr$$

$$= bP'(b)Q(b) - aP'(a)Q(a) - PrQ'\Big|_a^b$$

B.C.

$$+ \underline{\int_a^b \left(P(rQ')' - \frac{\nu^2}{r}PQ\right)dr} \qquad \boxed{= \int_a^b P(LQ)\,dr}$$

$$bP'(b)Q(b) - aP'(a)Q(a) - bP(b)Q'(b) + aP(a)Q'(a)$$

$$= b\beta \cancel{P(b)Q(b)} - a\alpha\cancel{P(a)Q(a)} - b\cancel{\beta P(b)}Q(b) + a\alpha\cancel{P(a)Q(a)}$$

$$= 0$$

You can use this blank page.

(b) From (a), $\displaystyle\int_a^b \underbrace{(LP)}_{LP=\lambda_1 rP} Q\,dr = \int_a^b P\underbrace{(LQ)}_{LQ=\lambda_2 rQ}\,dr$

$$\Longrightarrow \quad \lambda_1 \int_a^b PQr\,dr = \lambda_2 \int_a^b PQr\,dr$$

$$\underbrace{\cancel{(\lambda_1-\lambda_2)}}_{\neq 0} \int_a^b PQr\,dr = 0$$

$$\Longrightarrow \quad \boxed{\int_a^b PQr\,dr = 0}$$

(c) Let $LP = \lambda r P$

$$(LP)P = \lambda r P^2$$

$$\underbrace{\int_a^b (LP)P}_{\int_a^b \left((rP')' - \frac{\nu^2}{r}P\right)P\,dr} = \lambda \int_a^b rP^2\,dr$$

$$\int_a^b \left((rP')' - \frac{\nu^2}{r}P\right)P\,dr = \int_a^b \left((rP')'P - \frac{\nu^2 P^2}{r}\right)dr$$

$$= rP'P\Big|_a^b - \int_a^b \left(r(P')^2 + \frac{\nu^2P^2}{r} \right) dr$$

You can use this blank page.

$$= bP'(b)P(b) - aP'(a)P(a) - \int_a^b \left(r(P')^2 + \frac{\nu^2P^2}{r} \right) dr$$

$$= \beta b P^2(b) - a\alpha P^2(a) - \int_a^b \left(r(P')^2 + \frac{\nu^2P^2}{r} \right) dr$$

Hence

$$\underbrace{\beta b P^2(b) - a\alpha P^2(a)}_{-ve} \underset{-ve}{-} \underbrace{\int_a^b \left(r(P')^2 + \frac{\nu^2P^2}{r} \right) dr}_{+ve} = \lambda \underbrace{\int_a^b rP^2 dr}_{+ve}$$

$$\Longrightarrow \quad \lambda \leq 0$$

If $\lambda = 0$, then $P(a) = 0$, $P(b) = 0$, $\int_a^b r(P')^2 + \frac{\nu^2 P^2}{r} dr = 0$

$$\Longrightarrow \quad P(r) \equiv 0 \quad (\text{Contradictory to } P \not\equiv 0)$$

$$\Longrightarrow \quad \boxed{\lambda < 0}$$