Hw 1 (1.1) #7

$$i_x + C v_t + G v = 0$$

$$v_x + L i_t + R i = 0$$

∂_x

∂_t

$$i_{xx} + C v_{tx} + G v_x = 0$$

$$v_{xt} + L i_{tt} + R i_t = 0$$

$$i_{xx} + C\left(-L i_{tt} - R i_t\right) + G\left(-L i_t - R i\right) = 0$$

$$\boxed{i_{xx} = CL\, i_{tt} + (RC + GL)\, i_t + GR\, i}$$

(1.1) #8 Let $i(x,t) = f(x,t)e^{at}$ $a = ?$

$$f_{xx}\cancel{e^{at}}$$

$$= CL\left[f_{tt}\cancel{e^{at}} + 2af_t\cancel{e^{at}} + a^2 f\cancel{e^{at}}\right]$$

$$+ (RC+GL)\left[f_t\cancel{e^{at}} + af\cancel{e^{at}}\right]$$

$$+ GR\left[f\cancel{e^{at}}\right]$$

$$= CL f_{tt} + \underbrace{(CL\,2a + RC + GL)}_{=0}f_t + \left(CLa^2 + (RC+GL)a + GR\right)f$$

$=0$ if $a = -\left(\dfrac{RC+GL}{2CL}\right)$

Hw 1 (1.2) #5

(a) $u_{xx} + u_{yy} = 0, \quad u(0,y) = u(1,y) = u(x,0) = 0$

$$u_n(x,y) = \sin(n\pi x)\sinh(n\pi y), \quad n = 1, 2, 3, \dots$$

$$u_{n\,xx} = -(n\pi)^2 \sin(n\pi x)\sinh(n\pi y)$$

$$u_{n\,yy} = (n\pi)^2 \sin(n\pi x)\sinh(n\pi y)$$

$+$

$= 0$

$$\sinh(z) = \frac{e^{z} - e^{-z}}{2}$$

$$\sinh'(z) = \frac{e^{z} + e^{-z}}{2}$$

$$\sinh''(z) = \frac{e^{z} - e^{-z}}{2} = \sinh(z)$$

$$u_n(0,y) = \sin(n\pi x)\sinh(n\pi y)\Big|_{x=0} = 0$$

$$u_n(1,y) = \sin(n\pi x)\sinh(n\pi y)\Big|_{x=1} = 0$$

$$u_n(x,0) = \sin(n\pi x)\sinh(n\pi y)\Big|_{y=0} = 0$$

(b) $u(x,1) = \sin 2\pi x - \sin 3\pi x$

$$u(x,y) = \sum_n c_n \sin n\pi x \sinh(n\pi y)$$

$$= C_2 \sin 2\pi x \sinh(2\pi y) + C_3 \sin 3\pi x \sinh(3\pi y)$$

$$\downarrow y = 1$$

$$= \underbrace{\left(C_2 \sinh(2\pi)\right)}_{=1} \sin 2\pi x + \underbrace{\left(C_3 \sinh(3\pi)\right)}_{-1} \sin 3\pi x$$

$$\underline{C_2 = \frac{1}{\sinh(2\pi)}} \qquad \underline{C_3 = -\frac{1}{\sinh(3\pi)}}$$

(c) $\tilde{u}_n(x,y) = \sin(n\pi x) \sinh(n\pi(1-y))$

Similarly, $\tilde{u}_{n\,xx} + \tilde{u}_{n\,yy} = 0$

$\tilde{u}_n(0,y) = \tilde{u}_n(1,y) = 0$

$\tilde{u}_n(x,1) = 0 \qquad (\sinh(0) = 0)$

(d) $\tilde{u}_n(x,y) = C_n \sin n\pi x \sinh(n\pi(1-y))$

$\tilde{u}_1(x,y) = C_1 \sin\pi x \sinh(\pi(1-y))$

$\tilde{u}_1(x,0) = C_1 \sin\pi x \sinh(\pi)$

$= 2\sin\pi x$

$\Rightarrow C_1 = \dfrac{2}{\sinh(\pi)}$

(e) $u_{xx} + u_{yy} = 0,$

$u(0,y) = u(1,y) = 0$

$u(x,0) = 2\sin\pi x,$

$u(x,1) = \sin 2\pi x - \sin 3\pi x$

$$\boxed{u(x,y) = (b) + (d)}$$

$$= \sin 2\pi x \, \frac{\sinh(2\pi y)}{\sinh(2\pi)} - \sin 3\pi x \, \frac{\sinh(3\pi y)}{\sinh(3\pi)}$$

$$+ \frac{2 \sin \pi x \, \sinh(\pi(1-y))}{\sinh(\pi)}$$

<u>Note</u>: $(b)\Big|_{y=0} = 0$

Hence $u(x,0) = (d)\Big|_{y=0} = 2 \sin \pi x$

$(d)\Big|_{y=1} = 0$

Hence $u(x,1) = (b)\Big|_{y=1}$

$= \sin 2\pi x - \sin 3\pi x$

(1.3) #4

$$u_t = \frac{1}{10} u_{xx}, \qquad u_x(0,t) = u_x(1,t) = 0$$

$$u(x,0) = 3 - 4\cos 2x \qquad 0 < x < \pi$$

$$u(x,t) = a_0(t) + a_1(t)\cos 2x$$

satisfies

$$t=0 \Rightarrow \quad a_0(0) = 3, \quad a_1(0) = -4$$

$$u_t = \dot{a}_0(t) + \dot{a}_1(t)\cos 2x$$

$$u_{xx} = -4a_1(t)\cos 2x$$

Hence $\dot{a}_0 + \dot{a}_1 \cos 2x = \frac{1}{10}\left(-4a_1 \cos 2x\right)$

$$\dot{a}_0 = 0, \quad \Rightarrow \quad a_0(t) = C_0 \text{ a constant}$$
$$= 3$$

$$\dot{a}_1 = -\frac{2}{5}a_1 \Rightarrow a_1(t) = C_1 e^{-\frac{2}{5}t}$$
$$= -4e^{-\frac{2}{5}t}$$

Hence

$$\boxed{u(x,t) = 3 - 4e^{-\frac{2}{5}t}\cos 2x}$$

$$u(x,t) - 3 = -4e^{-\frac{2}{5}}\cos 2x$$

$$|u(x,t) - 3| = |-4e^{-\frac{2}{5}t}\cos 2x|$$

$$\leq 4e^{-\frac{2}{5}t} \leq 10^{-4}$$

$$\Rightarrow \quad 4 \times 10^{4} \leq e^{\frac{2}{5}t}$$

$$\Rightarrow \quad t \geq \frac{5}{2}\log(4 \times 10^{4})$$

$$\simeq 26.49$$

t_0

Add. Prob

$$\ddot{X} + 2\dot{X} = AX,$$

$$X(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \dot{X}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$A = \begin{pmatrix} -5 & 2 \\ -6 & 2 \end{pmatrix}$$

Note: A has 2 eigenvalues & eigenvectors:

$$\begin{pmatrix} -5 & 2 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \lambda_1, X_1$$

$$\begin{pmatrix} -5 & 2 \\ -6 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = (-2)\begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \lambda_2, X_2$$

Write $\boxed{X(t) = c_1(t)\, X_1 + c_2(t)\, X_2}$

? ?

$$\left(\ddot{c}_1 X_1 + \ddot{c}_2 X_2\right) + 2\left(\dot{c}_1 X_1 + \dot{c}_2 X_2\right)$$
$$= A\left(c_1 X_1 + c_2 X_2\right)$$
$$= c_1 \lambda_1 X_1 + c_2 \lambda_2 X_2$$
$$= -1\, c_1 X_1 - 2 c_2 X_2$$

Hence $\ddot{c}_1 + 2\dot{c}_1 + c_1 = 0$

$$\ddot{c}_2 + 2\dot{c}_2 + 2c_2 = 0$$

$r^2 + 2r + 1 = 0 \Rightarrow r = -1, -1$

$$c_1(t) = ae^{-t} + bte^{-t}$$

$r^2 + 2r + 2 = 0 \Rightarrow r = \dfrac{-2 \pm \sqrt{4-8}}{2} = -1 - i$

$$c_2(t) = e^{-t}\left(c \cos t + d \sin t\right)$$

Hence

$$X(t) = \left(ae^{-t} + bte^{-t}\right) X_1 + e^{-t}\left(c \cos t + d \sin t\right) X_2$$

$$X(0) = \underline{aX_1 + cX_2}$$

$$\dot{X}(t) = \left(-ae^{-t} + be^{-t} - bte^{-t}\right)X_1$$

$$- e^{-t}\left(c\cos t + d\sin t\right)X_2$$

$$+ e^{-t}\left(-c\sin t + d\cos t\right)X_2$$

$$\dot{X}(0) = \underline{(-a+b)X_1 + (-c+d)X_2}$$

$$a\begin{pmatrix}1\\2\end{pmatrix} + c\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \Rightarrow c=1,\quad a=-1$$

$$(-a+b)\begin{pmatrix}1\\2\end{pmatrix} + (-c+d)\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}$$

$$\Rightarrow -c+d = 3 \Rightarrow d=4$$

$$\Rightarrow -a+b = -5 \Rightarrow b=-6$$

$$\boxed{X(t) = \left(-e^{-t} - 6te^{-t}\right)\begin{pmatrix}1\\2\end{pmatrix} + e^{-t}\left(\cos t + 4\sin t\right)\begin{pmatrix}2\\3\end{pmatrix}}$$

Hw2 (2.3) # 1

$$(2.17) \quad \frac{\theta}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\theta$$

$$\int$$

$$\Rightarrow \frac{\theta^2}{4} = C + \sum_{n=1}^{\infty} \frac{\cancel{(-1)^{n+1}}(-1)}{n^2} \cos n\theta \qquad (-1)^n$$

$$\frac{\theta^2}{4} = C + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\theta$$

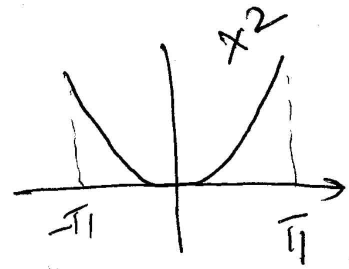
$$\int_{-\pi}^{\pi}$$

$$2\int_0^{\pi} \frac{\theta^2}{4} d\theta = C(2\pi) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cancel{\int_{-\pi}^{\pi} \cos n\theta \, d\theta} \qquad 0$$

$$\frac{2}{12}\pi^3 = 2\pi C \Rightarrow C = \frac{\pi^2}{12}$$

Hence $$\boxed{\theta^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\theta}$$

(2.3) #2



4. You are given the following Fourier series representation:

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx, \quad -\pi \le x \le \pi$$

Prove that

(a) $x^3 - \pi^2 x = 12 \sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n^3}$ for $-\pi \le x \le \pi$.

(b) $x^4 - 2\pi^2 x^2 = 48 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos nx}{n^4} - \frac{7\pi^4}{15}$ for $-\pi \le x \le \pi$.

(c) $\pi^4 = \frac{720}{7} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$

(a) integrate $\Rightarrow$ $\quad \frac{x^3}{3} = \frac{\pi^2 x}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n^3} + C$

Set $x=0 \Rightarrow C=0$

So $\quad x^3 - \pi^2 x = 12 \sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n^3}$

(b) integrate again: $\Rightarrow$

?

$$\frac{x^4}{4} - \frac{\pi^2 x^2}{2} = -12 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^4} + C$$

$$\int_{-\pi}^{\pi} \left(\frac{x^4}{4} - \frac{\pi^2 x^2}{2}\right) dx = \int_{-\pi}^{\pi} \left(\cancel{-12 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^4}}^{\,0} + c \right) dx$$

$$\cancel{2} \times \left[\frac{\pi^5}{20} - \frac{\pi^5}{6} \right] = C \times \cancel{2}\pi \qquad C = \frac{\pi^4}{20} - \frac{\pi^4}{6}$$

$$= \frac{-7\pi^4}{60}$$

This is a scrap paper.

So $x^4 - 2\pi^2 x^2 = 48 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos nx}{n^4} - \frac{7\pi^4}{15}$

(c) Set $x = 0$

$$\frac{7\pi^4}{15} = 48 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$$

$$\boxed{\pi^4 = \frac{720}{7} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}}$$

Set $x = \pi \Rightarrow$

$$\pi^4 - 2\pi^4 = 48 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cancel{\cos n\pi}^{(-1)^n}}{n^4} - \frac{7\pi^4}{15}$$

$$\left(\frac{7}{15} - 1\right)\pi^4 = 48 \sum_{n=1}^{\infty} \left(-\frac{1}{n^4}\right)$$

Hence $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{8}{15(48)} = \frac{\pi^4}{90}}$

(2.3) #3

$$|\theta| = \begin{cases} \theta & \theta > 0 \\ -\theta & \theta < 0 \end{cases}$$

$$\underbrace{\theta(\pi - |\theta|)}_{\text{odd}} = \underbrace{\frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)\theta}{(2n-1)^3}}_{\text{odd}}$$

$$\int_0^\theta \phi(\pi - \phi)\, d\phi \qquad (0 < \theta < \pi)$$

$$= \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \int_0^\theta \sin(2n-1)\phi\, d\phi$$

$$L.H.S. = \frac{\pi\theta^2}{2} - \frac{\theta^3}{3}$$

$$R.H.S. = -\frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \cos(2n-1)\phi \Big|_0^\theta$$

$$= -\frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \cos(2n-1)\theta$$

$$+ \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \leftarrow C = ?$$

Hence

$$\frac{\pi\theta^2}{2} - \frac{\theta^3}{3} = -\frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)\theta}{(2n-1)^4} + C$$

$$\int_0^{\pi}\left(\frac{\pi\theta^2}{2}-\frac{\theta^3}{3}\right)d\theta$$

$$= -\frac{8}{\pi}\sum_{n=1}^{\infty}\underbrace{\cancel{\int_0^{\pi}\frac{\cos(2n-1)\theta}{(2n-1)^4}\,d\theta}}_{=0} + C\int_0^{\pi}d\theta$$

$$\frac{\pi^4}{6}-\frac{\pi^4}{12}=\pi C \implies C=\frac{\pi^3}{12}$$

Hence for $0<\theta<\pi$

$$\frac{\pi\theta^2}{2}-\frac{\theta^3}{3}=\frac{\pi^3}{12}-\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\theta}{(2n-1)^4}$$

$$\implies \boxed{\sum_{n=1}^{\infty}\frac{\cos(2n-1)\theta}{(2n-1)^4}=\frac{\pi\theta^3}{24}-\frac{\pi^2\theta^2}{16}-\frac{\pi^4}{96}}$$

(for $0<\theta<\pi$)

Note: original function is odd. Hence its integral is even. Thus

$$\boxed{\sum_{n=1}^{\infty}\frac{\cos(2n-1)\theta}{(2n-1)^4}=\frac{\pi|\theta|^3}{24}-\frac{\pi^2\theta^2}{16}-\frac{\pi^4}{96}}$$

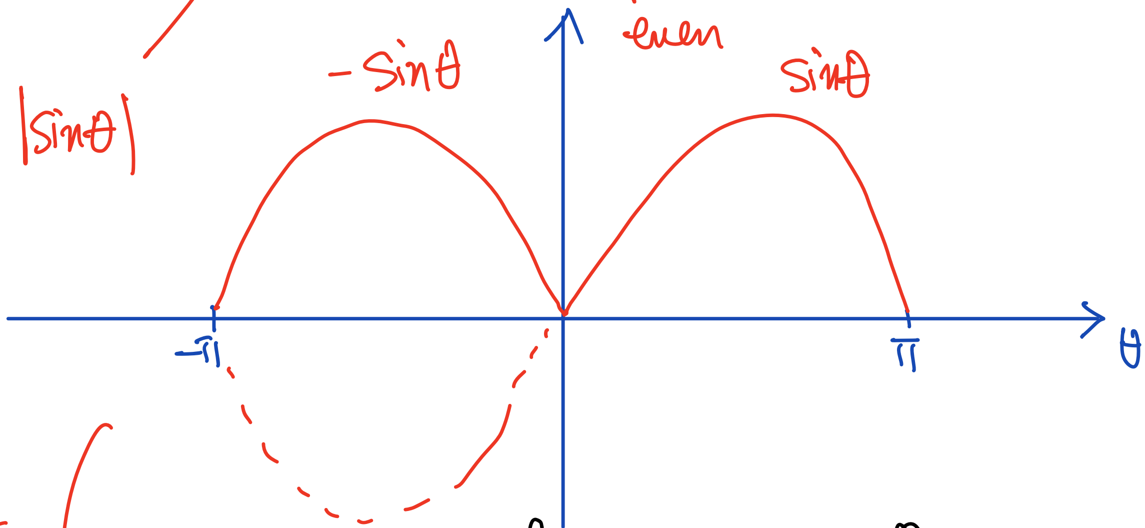
(for $-\pi<\theta<\pi$)

(2.3) #4

$$\boxed{\sin\theta = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2n\theta}{4n^2-1}} \quad 0<\theta<\pi$$

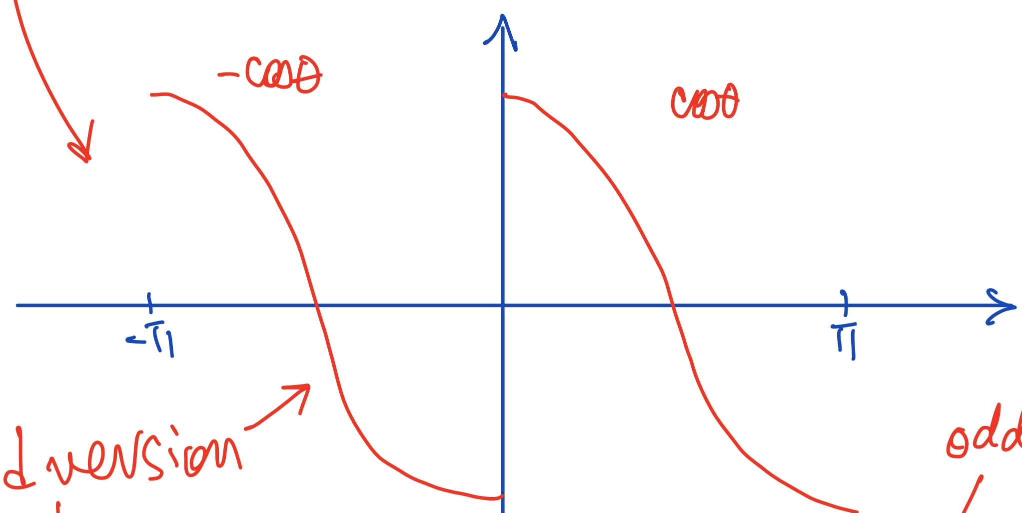
even

$|\sin\theta|$



$\frac{d}{d\theta}$

$$\frac{d}{d\theta}|\sin\theta| = \begin{cases} \cos\theta & 0<\theta<\pi \\ -\cos\theta & -\pi<\theta<0 \end{cases}$$



odd version

odd

$$\boxed{\cos\theta = \frac{d}{d\theta}\sin\theta = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{2n\sin(2n\theta)}{4n^2-1}} \quad ①$$

$$\cos\theta = -\int_{\pi/2}^{\theta} \sin\phi \, d\phi$$

$$= -\int_{\pi/2}^{\theta} \left(\frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2n\phi}{(4n^2-1)} \right) d\phi$$

$$= -\frac{2}{\pi}\left(\theta - \frac{\pi}{2}\right) + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2n\phi}{2n(4n^2-1)} \Bigg|_{\pi/2}^{\theta}$$

$$= -\frac{2\theta}{\pi} + 1 + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{2n(4n^2-1)}$$

Hence

$$\underbrace{\cos\theta + \frac{2\theta}{\pi} - 1}_{} = \underbrace{\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{2n(4n^2-1)}}_{\text{odd}}$$

Let

$$f(\theta) = \begin{cases} \cos\theta + \dfrac{2\theta}{\pi} - 1 & 0 < \theta < \pi \\[2ex] -\cos\theta + \dfrac{2\theta}{\pi} + 1 & -\pi \;\; \theta < 0 \end{cases}$$

this is an odd function : $f(-\theta) = -f(\theta)$

Hence

$$\begin{Bmatrix}\cos\theta\\ -\cos\theta\end{Bmatrix} = -\frac{2\theta}{\pi} + \begin{Bmatrix}1\\ -1\end{Bmatrix} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n\theta)}{2n(4n^2-1)}$$

Use Table 1 #1 or (2.17)   Table 1 #6

$$= -\frac{2}{\pi}\left[2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\theta\right] + \frac{4}{\pi}\sum_{1}^{\infty}\frac{\sin(2n-1)\theta}{2n-1} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2n\theta}{2n(4n^2-1)}$$

$$= -\frac{4}{\pi}\left[\cancel{\sin\theta} - \frac{\sin 2\theta}{2} + \cancel{\frac{\sin 3\theta}{3}} - \frac{\sin 4\theta}{4} + \cdots\right]$$

$$+ \frac{4}{\pi}\left[\cancel{\frac{\sin\theta}{1}} + \cancel{\frac{\sin 3\theta}{3}} + \cdots\right]$$

$$+ \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n\theta)}{2n(4n^2-1)}$$

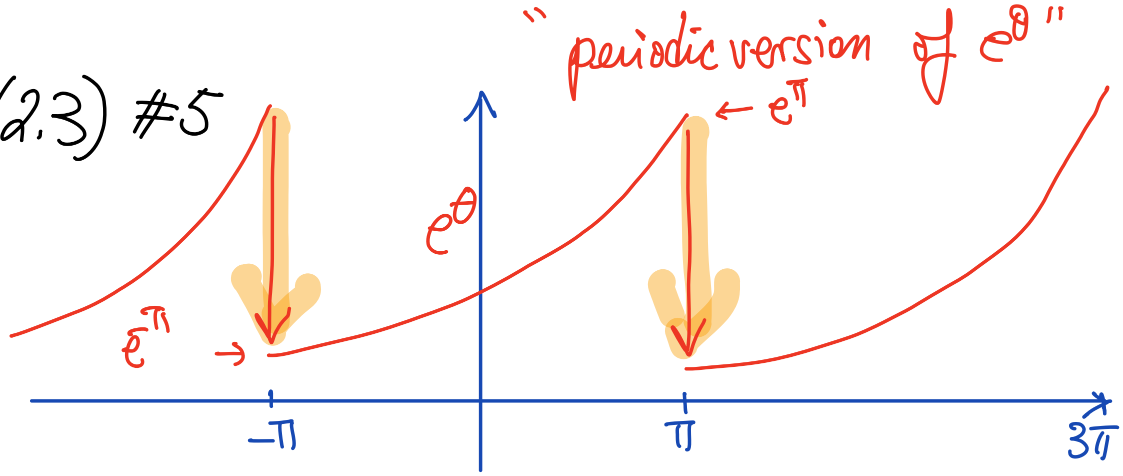
$$= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2n\theta}{2n} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin 2n\theta}{2n(4n^2-1)}$$

$$= \frac{4}{\pi} \sum_{n=1}^{\infty} \left(\frac{1}{2n} + \frac{1}{2n(4n^2-1)} \right) \sin 2n\theta$$

$$= \boxed{\frac{4}{\pi} \sum_{n=1}^{\infty} \left(\frac{2n}{4n^2-1} \right) \sin 2n\theta}$$

same as (1)

(2.3) #5



$$e^\theta = \sum_{n=1}^{\infty} c_n e^{in\theta} \qquad (-\pi < \theta < \pi)$$

$$\frac{d}{d\theta}$$

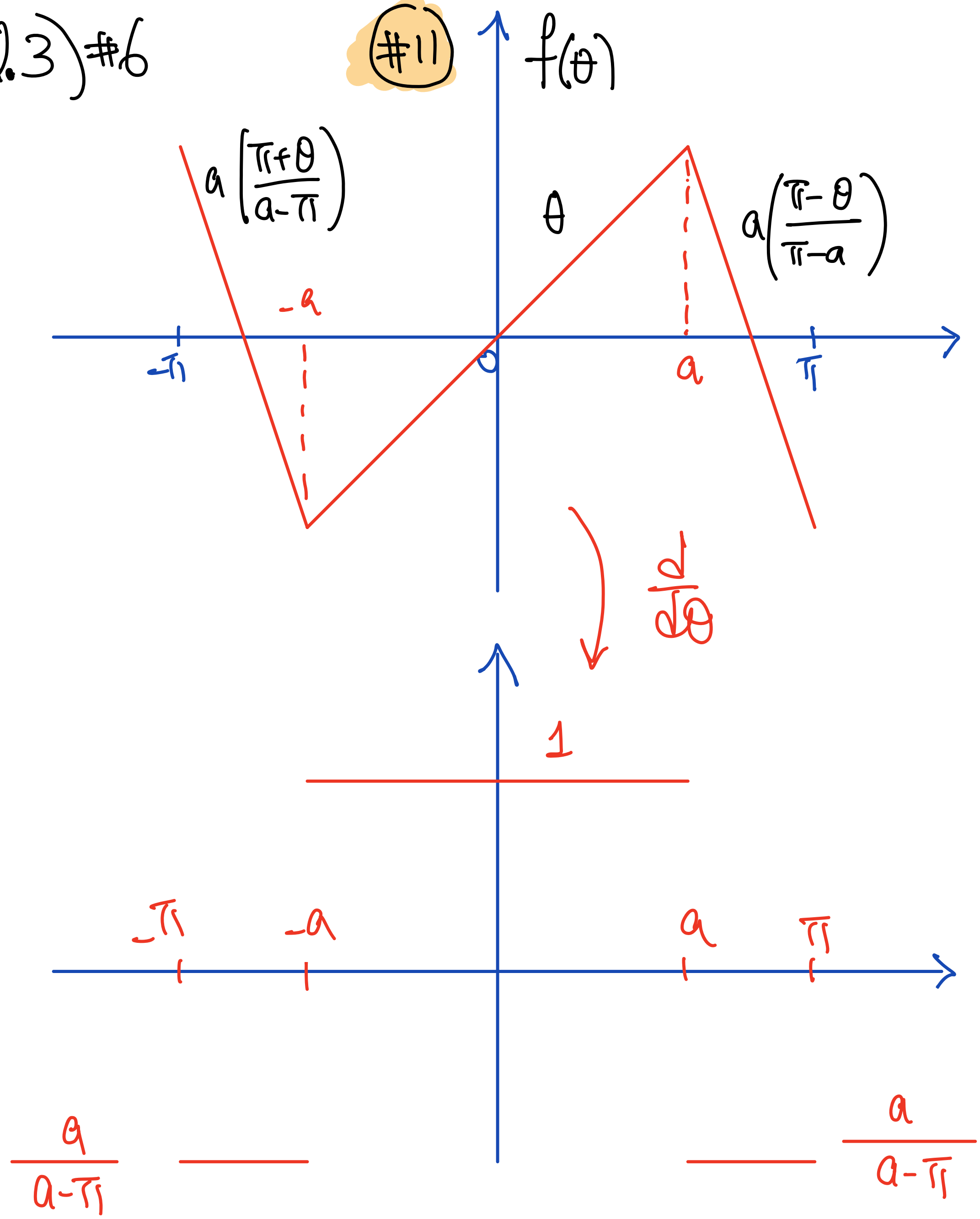
$$e^\theta + \underbrace{\boxed{\sum_{n=-\infty}^{\infty} \left(e^{-\pi} - e^{\pi}\right) \delta_{(2n-1)\pi}(\theta)}}_{\text{delta function due to the discontinuity of } e^\theta \text{ across } \pi,\, 3\pi,\, \ldots} = \sum_{n=1}^{\infty} in\, c_n e^{in\theta}$$

i.e.

$$\boxed{\sum_{n=-\infty}^{\infty} \left(e^{-\pi} - e^{\pi}\right) \delta_{(2n-1)\pi}(\theta) = \sum_{n=1}^{\infty} (in-1)\, c_n e^{in\theta}}$$

(2.3) #6

#11



$$\left(\left[\frac{d}{d\theta} (11) \right] + \frac{a}{\pi - a} \right) \times \frac{(2a)^{-1}}{1 + \frac{a}{\pi - a}} = (12)$$

$$(12) = \frac{\pi - a}{2\pi a}\left(\frac{d(11)}{d\theta} + \frac{a}{\pi - a}\right)$$

$$= \frac{\pi - a}{2\pi a}\frac{d(11)}{d\theta} + \frac{1}{2\pi}$$

$$\frac{1}{2\pi} + \frac{1}{\pi}\sum_{1}^{\infty}\frac{\sin na}{na}\cos(n\theta)$$

$$\parallel$$

$$= \frac{\pi - a}{2\pi a}\left[\frac{2}{\pi - a}\sum_{1}^{\infty}\frac{\sin na}{n}\cos n\theta\right] + \frac{1}{2\pi}$$

HW3 Add. Problem

(1) $$F_N(\theta) = \frac{1}{N+1}\left(D_0(\theta) + D_1(\theta) + \cdots + D_N(\theta)\right)$$

$$= \frac{1}{N+1}\left[\frac{\sin\frac{\theta}{2} + \sin\left(\frac{3\theta}{2}\right) + \sin\left(\frac{5\theta}{2}\right) + \cdots + \sin\left(\frac{2N+1}{2}\theta\right)}{2\pi \sin\theta/2}\right]$$

$$\left(\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}\right)$$

$$= \frac{1}{2\pi(N+1)}\left[\frac{e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}} + e^{\frac{3i\theta}{2}} - e^{-\frac{3i\theta}{2}} + e^{\frac{5i\theta}{2}} - e^{-\frac{5i\theta}{2}} + \cdots + \left(e^{\frac{2N+1}{2}i\theta} - e^{-\frac{2N+1}{2}i\theta}\right)}{2i \sin\theta/2}\right]$$

$$= \frac{1}{2\pi(N+1)\sin\frac{\theta}{2}}\left[\frac{\left(e^{\frac{i\theta}{2}} + e^{\frac{3i\theta}{2}} + \cdots + e^{\frac{2N+1}{2}i\theta}\right) - \left(e^{-i\theta/2} + e^{-\frac{i3\theta}{2}} + \cdots + e^{-i\frac{2N+1}{2}\theta}\right)}{2i}\right]$$

$$= \frac{1}{2\pi(N+1)\left(\sin\frac{\theta}{2}\right)(2i)}\left[e^{i\frac{\theta}{2}}\left(1+e^{i\theta}+e^{2i\theta}+\cdots+e^{iN\theta}\right) - e^{-i\frac{\theta}{2}}\left(1+e^{-i\theta}+e^{-2i\theta}+\cdots+e^{-iN\theta}\right)\right]$$

$$= \frac{1}{2\pi(N+1)\left(\sin\frac{\theta}{2}\right)(2i)}\left[e^{i\frac{\theta}{2}}\left(\frac{1-e^{i(N+1)\theta}}{1-e^{i\theta}}\right) - e^{-i\frac{\theta}{2}}\left(\frac{1-e^{-i(N+1)\theta}}{1-e^{-i\theta}}\right)\right]$$

$$= \frac{1}{2\pi(N+1)\left(\sin\frac{\theta}{2}\right)2i}\left[e^{i\left(\frac{N+1}{2}\right)\theta}\left(\frac{e^{-i\frac{N+1}{2}\theta}-e^{i\frac{N+1}{2}\theta}}{e^{-i\frac{\theta}{2}}-e^{i\frac{\theta}{2}}}\right) - e^{-i\left(\frac{N+1}{2}\right)\theta}\left(\frac{e^{i\frac{N+1}{2}\theta}-e^{-i\frac{N+1}{2}\theta}}{e^{i\frac{\theta}{2}}-e^{-i\frac{\theta}{2}}}\right)\right]$$

$$= \frac{1}{2\pi(N+1)\left(\sin\frac{\theta}{2}\right)\underline{2i}}\left[\underline{e^{i\left(\frac{N+1}{2}\right)\theta}}\,\frac{\sin\left(\frac{N+1}{2}\theta\right)}{\sin\frac{\theta}{2}} - \underline{e^{-i\frac{N+1}{2}\theta}}\,\frac{\sin\frac{N+1}{2}\theta}{\sin\theta/2}\right]$$

$$= \frac{1}{2\pi(N+1)} \frac{\sin^2\left(\frac{N+1}{2}\theta\right)}{\sin^2\frac{\theta}{2}}$$

(≥ 0)

$$\boxed{F_N(\theta) = \frac{1}{2\pi(N+1)} \left[\frac{\sin\left(\frac{N+1}{2}\theta\right)}{\sin\left(\frac{\theta}{2}\right)}\right]^2}$$

2 1. $\boxed{F_N(\theta) = \frac{D_0(\theta) + D_1(\theta) + \cdots + D_N(\theta)}{N+1}}$

$\underline{D_i(\theta) - 2\pi\text{-per}} \Rightarrow \underline{F_N - 2\pi\text{-per.}}$

2. $\underline{D_i - \text{even}} \Rightarrow \underline{F_N - \text{even}}$

3. $$\int_{-\pi}^{\pi} F_N(\theta)\,d\theta = \frac{1}{N+1}\int_{-\pi}^{\pi}\left(D_0(\theta) + \cdots + D_N(\theta)\right)d\theta$$

$$= \frac{1}{N+1}\left[1 + 1 + \cdots + 1\right]$$

$$\underline{= \frac{N+1}{N+1} = 1}$$

4. $$F_N(0) = \frac{1}{2\pi(N+1)} \left[\frac{\sin\left(\frac{N+1}{2}\theta\right)}{\sin\left(\frac{\theta}{2}\right)} \right]^2 \Bigg|_{\theta=0}$$

$$\left.\frac{\sin \frac{N+1}{2}\theta}{\sin\frac{\theta}{2}}\right|_{\theta=0} \overset{L'R}{=} \left.\frac{\frac{N+1}{2} \cos\frac{N+1}{2}\theta}{\frac{1}{2}\cos\frac{\theta}{2}}\right|_{\theta=0} = N+1$$

$$= \frac{1}{2\pi(N+1)}(N+1)^2 = \frac{N+1}{2\pi}$$

5.

$$F_N(\theta) = \frac{1}{2\pi(N+1)} \left[\frac{\overbrace{\sin\left(\frac{N+1}{2}\theta\right)}^{\sim \pm 1}}{\sin\left(\frac{\theta}{2}\right)} \right]^2$$

when $\theta \sim \pm\pi$

$$\cong \frac{1}{2\pi(N+1)} \frac{(\pm 1)^2}{\sin\frac{\pi}{2}} = \frac{1}{2\pi(N+1)} \ll 1 \quad (N \gg 1)$$

③

$$F_N(\theta) = \frac{1}{2\pi(N+1)}\left[\frac{\sin\left(\frac{N+1}{2}\theta\right)}{\sin\left(\frac{\theta}{2}\right)}\right]^2 < \varepsilon$$

$$\left(\sin\frac{N+1}{2}\theta\right)^2 \leq 2\pi(N+1)\left(\sin\frac{\theta}{2}\right)\varepsilon$$

$$\left[\sin\left(\frac{N+1}{2}\theta\right)\right]^2 \leq 1 \leq 2\pi(N+1)\left(\sin\frac{\delta}{2}\right)\varepsilon \leq 2\pi(N+1)\left(\sin\frac{\theta}{2}\right)\varepsilon$$

$|\sin| \leq 1$

$\delta < |\theta|$

$\sin\theta/2$ is an increasing function for $0<\theta<\pi$

$$N+1 \geq \frac{1}{2\pi\left(\sin\frac{\delta}{2}\right)\varepsilon}, \qquad N \geq \frac{1}{2\pi\left(\sin\frac{\delta}{2}\right)\varepsilon} - 1$$

$\varepsilon = 10^{-6}$, $\delta = 10^{-5}$,

$$\boxed{N \geq \frac{1}{2\pi\left(\sin\frac{10^{-5}}{2}\right)10^{-6}} - 1}$$

$$\cong 3\times 10^{10}$$

Hw 4

(3.3) #1 $\quad f_n \to f$ in norm means

$$\|f_n - f\| \to 0$$

$$|\langle f_n, g\rangle - \langle f, g\rangle| = |\langle f_n - f, g\rangle|$$

$$\overset{C.S.}{\leq} \underbrace{\|f_n - f\|}_{\downarrow 0} \|g\|$$

(3.2) #2

$$|a| \leq b \iff -b < a < b$$

$$\big|\|f\| - \|g\|\big| \leq \|f-g\|$$

$$\iff -\|f-g\| \leq \|f\| - \|g\| \leq \|f-g\|$$

true? Yes $\qquad$ true? Yes

$$\iff \|g\| \overset{?}{\leq} \|f\| + \|f-g\|$$

$$\left(\begin{aligned} \|g\| &= \|g-f+f\| \\ &\leq \|g-f\| + \|g\| \end{aligned} \right)$$

$$\iff \|f\| \overset{?}{\leq} \|f-g\| + \|g\|$$

$$\left(\begin{aligned} &\text{By } \Delta\text{-Ineq:} \\ &\|f\| = \|f-g+g\| \\ &\qquad \leq \|f-g\| + \|g\| \end{aligned} \right)$$

(3.4) #2 $\quad f_0(x)=1, \quad f_1(x)=ax+b$

$$f_2(x)=Ax^2+Bx+C$$

$$\langle f_0, f_1\rangle = \int_0^\infty 1\,(ax+b)\,e^{-x}\,dx$$

$$= a\left(\int_0^\infty xe^{-x}\,dx\right) + b\left(\int_0^\infty e^{-x}\,dx\right) = 0 \quad 1$$

$$\langle f_0, f_2\rangle = \int_0^\infty 1\,(Ax^2+Bx+C)\,e^{-x}\,dx$$

$$= A\left(\int_0^\infty x^2e^{-x}\right) + B\left(\int_0^\infty xe^{-x}\,dx\right) + C\left(\int_0^\infty e^{-x}\,dx\right)$$

$$= 0 \quad 2$$

$$\langle f_1, f_2\rangle = \int_0^\infty (ax+b)(Ax^2+Bx+C)\,e^{-x}\,dx = 0$$

$$= aA\left(\int_0^\infty x^3e^{-x}\,dx\right) + (aB+bA)\left(\int_0^\infty x^2e^{-x}\,dx\right)$$

$$+ (aC+bB)\left(\int_0^\infty xe^{-x}\,dx\right) + bC\left(\int_0^\infty e^{-x}\,dx\right)$$

$$= 0 \quad 3$$

$$\|f_1\|^2 = \int_0^\infty (ax+b)^2 e^{-x}\,dx$$

$$= a^2\left(\int_0^\infty x^2 e^{-x}\,dx\right) + 2ab\left(\int_0^\infty x e^{-x} e^{-x}\,dx\right) + b^2\left(\int_0^\infty e^{-x}\,dx\right) = 1 \quad (4)$$

$$\|f_2\|^2 = \int_0^\infty (Ax^2+Bx+C)^2 e^{-x}\,dx$$

$$= A^2\int_0^\infty x^4 e^{-x}\,dx + B^2\int_0^\infty x^2 e^{-x}\,dx + C^2\int_0^\infty e^{-x}\,dx$$

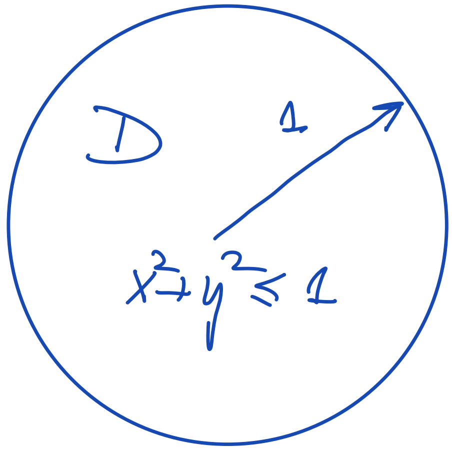
$$+ 2AB\int_0^\infty x^3 e^{-x}\,dx + 2BC\int_0^\infty x e^{-x}\,dx + 2AC\int_0^\infty x^2 e^{-x}\,dx \quad (5)$$

5 unknowns: $a, b, A, B, C,$

5 equations. (1), (2), (3), (4), (5)

(Note: $\|f_0\|^2 = \int_0^\infty 1^2 e^{-x}\,dx = 1$)

(3.4) #3



$$f_n(x,y) = (x+iy)^n$$

$$x+yi = re^{i\theta}$$

$$\overline{x+yi} = re^{-i\theta}$$

$$\langle f_n, f_m \rangle = \langle (x+iy)^n, (x+iy)^m \rangle_D$$

$$= \iint_D (x+iy)^n \, \overline{(x+iy)^m} \, dx\,dy$$

$$= \int_0^1 \int_0^{2\pi} r^n e^{in\theta} r^m e^{-im\theta} \, \underline{r} \, d\theta \, dr$$

$(dx\,dy = r\,dr\,d\theta)$

$$= \int_0^1 \underbrace{\int_0^{2\pi} (e^{in\theta} e^{-im\theta}) \, d\theta}_{=0 \quad (n \neq m)} \, r^{n+m+1} \, dr \quad = 0$$

$$\|f_n\|^2 = \int_0^1 \int_0^{2\pi} r^{2n} r \, dr \, d\theta = \frac{2\pi}{2n+2}$$

$(m=n)$

(3.4) #7 $\quad f(x) = x$

(a) $P_\pi f$

$$= \frac{\langle x, 1\rangle}{\|1\|^2}\,1 + \frac{\langle x, \cos x\rangle}{\|\cos x\|^2}\cos x + \frac{\langle x, \cos 2x\rangle}{\|\cos 2x\|^2}\cos 2x$$

$$\int_0^\pi x\cdot 1\,dx \qquad \int_0^\pi 1^2\,dx$$

(b) $$P_\pi f = \frac{\langle x, \sin x\rangle}{\|\sin x\|^2}\sin x + \frac{\langle x, \sin 2x\rangle}{\|\sin 2x\|^2}\sin 2x$$

(c) $$P_\pi f = \frac{\langle x, \cos x\rangle}{\|\cos x\|^2}\cos x + \frac{\langle x, \sin x\rangle}{\|\sin x\|^2}\sin x$$