Hw5

(3.5) #1

$$Lf = (rf')' + pf \quad \text{on } [a,b]$$

$$f(b) = c f(a), \quad f'(b) = c' f'(a)$$

$$\int_a^b (Lf) g \, dx \overset{?}{=} \int_a^b f (Lg) \, dx$$

$$\| $$

$$\int_a^b \left((rf')' + pf \right) g \, dx$$

$$= rf'g \Big|_a^b - \int_a^b f' r g' \, dx + \int_a^b p f g \, dx$$

$$= b f'(b) g(b) - a f'(a) g(a) - f r g' \Big|_a^b$$

$$+ \int_a^b f (rg')' + p f g \, dx$$

$$b f'(b) g(b) - a f'(a) g(a) - b f(b) g'(b) + a f(a) g'(a)$$

$$= b c' f'(a) c g(a) - a f'(a) g(a) - b c f(a) c' g'(a)$$
$$+ a f(a) g''(a)$$

$$= (bcc' - a) f'(a) g(a) - (bcc' - a) f(a) g'(a)$$

$$= \boxed{(bcc' - a)} f'(a) g(a) = 0 \qquad \boxed{bcc' - a = 0}$$

(3.5) #2 $\quad f''(x) + \lambda f(x) = 0$

$$f'(0) = \alpha f(0), \quad f'(\ell) = \beta f(\ell)$$

(1) $\alpha > 0 > \beta$ $\quad$ Suppose $\lambda < 0$

$$\Rightarrow \int_0^\ell f''(x) f(x) + \lambda f(x)^2 = 0$$

$$f'(x) f(x)\Big|_0^\ell + \int_0^\ell (\underbrace{-f'(x)^2}_{<0} + \underbrace{\lambda f(x)^2}_{<0})\,dx = 0$$

$$\| $$

$$f'(\ell)f(\ell) - f'(0)f(0) = \underbrace{\beta}_{<0} f(\ell)^2 \underbrace{- \alpha}_{<0} f(0)^2$$

?!

(2) $\beta > \alpha > 0$ or $0 > \beta > \alpha$. Suppose $\lambda < 0$

$$f''(x) + \lambda f(x) = 0 \quad (r^2 + \lambda = 0, \quad r = \pm\sqrt{-\lambda})$$

$$\Rightarrow \quad f(x) = A e^{\sqrt{-\lambda}x} + B e^{-\sqrt{-\lambda}x}$$

$$f'(x) = \sqrt{-\lambda}\left[A e^{\sqrt{-\lambda}x} - B e^{-\sqrt{-\lambda}x}\right]$$

$$f'(0) = \alpha f(0) \implies \sqrt{-\lambda}\,[A - B] = A + B$$

$$f'(l) = \beta f(l) \implies \sqrt{-\lambda}\left[A e^{\sqrt{-\lambda}\,l} - B e^{-\sqrt{-\lambda}\,l}\right]$$

$$= A e^{\sqrt{-\lambda}\,l} + B e^{-\sqrt{-\lambda}\,l}$$

$$\implies (\sqrt{-\lambda} - 1)A = (1 + \sqrt{-\lambda})B$$

$$(\sqrt{-\lambda} - 1)e^{\sqrt{-\lambda}\,l} A = (1 + \sqrt{-\lambda})\, e^{-\sqrt{-\lambda}\,l} B$$

if $\sqrt{-\lambda} - 1 \neq 0$, then

$e^{-\sqrt{-\lambda}\,l} = e^{\sqrt{-\lambda}\,l}$ not possible unless $\lambda = 0$

if $\sqrt{-\lambda} - 1 = 0$, i.e. $\lambda = -1$, then $B = 0$

and A can be arbitrary, and

$$\boxed{f(x) = e^{x} \text{ is an eigenfunction}}$$

$$\boxed{\lambda = -1 \text{ is the only -ve eigenvalue}}$$

(3.5) #12

$$(rf')' + pf + \lambda f = 0$$
$$f(1) = f(b) = 0, \quad b > 1$$

(a) $$\int_1^b \left((rf')' + pf + \lambda f\right) f \, dx = 0$$

$$rf'f\Big|_1^b + \int_1^b -r(f')^2 + pf^2 + \lambda f^2 \, dx = 0$$

$$\| $$

$$r(b)f'(b)\underbrace{\cancel{f(b)}}_{0} - r(1)f'(1)\underbrace{\cancel{f(1)}}_{0}$$

Hence $$\lambda \int_1^b |f|^2(x)\, dx = \int_1^b r(f')^2 - p(f)^2 \, dx$$

(b) If $p \le C \Rightarrow -p \geq -C$

then $$\lambda \int_1^b |f|^2 dx \geq -\int_1^b p|f|^2 dx$$
$$\geq -C\int_1^b |f|^2 dx$$

i.e. $$\boxed{\lambda \geq -C}$$

(c) $\alpha \geq 0, \beta \leq 0$

$$f'(a) = \alpha f(a), \quad f'(b) = \beta f(b)$$

Then

$$r f' f \Big|_a^b + \int_a^b - r(f')^2 + p f^2 + \lambda f^2 \, dx = 0$$

$$\Vert$$

$$r(b) f'(b) f(b) - r(a) f'(a) f(a)$$

$$r(b) \beta f(b)^2 - r(a) \alpha f(a)^2$$

Hence

$$\lambda \int_a^b |f|^2 dx = \int_a^b \overset{+ve}{r} (f')^2 - p(f)^2$$

$$- \underbrace{\beta}_{+ve} r(b) f(b)^2 + \underbrace{\alpha}_{+ve} r(a) f(a)^2$$

$$\geq - \int_a^b p |f|^2 dx \geq -C \int_a^b |f|^2 dx$$

$$\Rightarrow \boxed{\lambda > -C}$$

(4.2) #3 $\quad u_t = u_{xx} \quad 0 \le x \le l$

$u_x(0,t) = 0, \quad u_x(l,t) = A$

① Let $g(x) = cx^2$. Then $g_x(x) = 2cx$

$g_x(0) = 0, \quad g_x(l) = 2cl = A \Rightarrow c = \frac{A}{2}$

$g(x) = \frac{A}{2}x^2$

② Let $u(x,t) = \tilde{u}(x,t) + g(x)$

Then $\quad \tilde{u}_t = \tilde{u}_{xx} + \tilde{g}_{xx}$

i.e. $\quad \tilde{u}_t = \tilde{u}_{xx} + A$

and $\quad \tilde{u}_x(0,t) = 0$ and $\tilde{u}_x(l,t) = 0$

$\varphi_n(x) = \cos\left(\frac{n\pi x}{l}\right), \quad n = 0, 1, 2, \dots$

Let
$$\boxed{\begin{aligned} \tilde{u}(x,t) &= \sum_{n=0}^{\infty} C_n(t)\varphi_n(x) \\ A &= \sum_{n=0}^{\infty} b_n(t)\varphi_n(t) \end{aligned}}$$

$$\Rightarrow \sum_{n=0}^{\infty} \dot{c}_n(t)\varphi_n = \sum_{n=0}^{\infty} c_n(t)\lambda_n \varphi_n(t) + \sum_{n=0}^{\infty} b_n \varphi_n$$

$$\lambda_n = -\frac{n^2\pi^2}{\ell^2}$$

Variation of Parameter or integrating factor

Hence $$\boxed{\dot{c}_n(t) = +\lambda_n c_n(t) + b_n}$$

$$\boxed{c_n(t) = c_n(0)e^{\lambda_n t} + \int_0^t e^{\lambda_n(t-s)} b_n \, ds}$$

$$= \begin{cases} c_0(0) + b_0 t & n=0 \quad (\lambda_0 = 0) \\ c_n(0)e^{\lambda_n t} + \frac{b_n}{\lambda_n} e^{\lambda_n t}, & n \geq 1 \quad (\lambda_n \neq 0) \end{cases}$$

$\begin{cases} c_n(0) \text{ comes from } \tilde{u}(x,0) = u(x,0) - g(x) \\ b_n \text{ comes from } A. \end{cases}$

$$\boxed{u(x,t) = \tilde{u}(x,t) + g(x)}$$

(4.2) #7 $\quad u_t = k u_{xx} + R$

$$u_x(0,t) = u_x(l,t) = 0$$

(a) There is NO steady state, $g(x)$. For otherwise,

$$\boxed{0 = k g_{xx} + R}$$

$$\int_0^l \left(0 = k g_{xx} + R \right) dx$$

$$0 = k(\underbrace{\cancel{g_x(l)}}_{0} - \underbrace{\cancel{g_x(0)}}_{0}) + Rl$$

$$0 = Rl \quad !!$$

(b) (M1) Let $u(x,t) = \sum_{n=0}^{\infty} c_n(t)\, \varphi_n(x)$

$$R = \sum_{n=0}^{\infty} b_n\, \varphi_n(x)$$

$$\left(\varphi_n(x) = \cos\left(\frac{n\pi x}{l}\right), \quad n = 0, 1, 2, \cdots \right)$$

Then
$$\sum_{n=0} \dot{c}_n(t)\varphi_n = \sum_{n=0} c_n(t)\lambda_n\varphi_n + \sum_{n=0} b_n$$

$$\Longrightarrow \qquad \dot{c}_n(t) = \lambda_n c_n(t) + b_n$$

$$c_n(t) = c_n(0)e^{\lambda_n t} + \int_0^t e^{\lambda_n(t-s)} b_n\, ds$$

(M2) Let $\underline{g(x,t) = Rt}$ (Note: not a steady state.)

Let $u = \tilde{u} + g$

$$(u_t = u_{xx} + R)$$

$$\tilde{u}_t + \underbrace{\cancel{g_t}}_{R} = \tilde{u}_{xx} + \underbrace{\cancel{g_{xx}}}_{0} + R$$

cancel

Hence $\tilde{u}_t = \tilde{u}_{xx}$

$$\left(\begin{array}{l} \tilde{u}_x(0,t) = \cancel{u_x(0,t)}^{\,0} + \cancel{g_x(0,t)}^{\,0} = 0 \\ \tilde{u}_x(\ell,t) = \cancel{u_x(\ell,t)} + \cancel{g_x(\ell,t)} = 0 \end{array}\right)$$

(4.2) #8 $\quad u_t = k u_{xx} \quad 0 \le x \le l$

$u_x(0,t) = 0$ and $u_x(l,t) + b u(l,t) = 0$

$(b > 0)$

① Find eigenfunctions: $\quad \varphi_{xx} = \lambda \varphi, \quad 0 < x < l$

$\varphi_x(0) = 0, \quad \varphi_x(l) + b\varphi(l) = 0$

Note: $\lambda < 0$ (why?)

$$\varphi(x) = A \cos(\sqrt{-\lambda}\, x) + B \sin(\sqrt{-\lambda}\, x)$$

$$\varphi_x(x) = -A\sqrt{-\lambda}\sin(\sqrt{-\lambda}\, x) + B\sqrt{-\lambda}\cos(\sqrt{-\lambda}\, x)$$

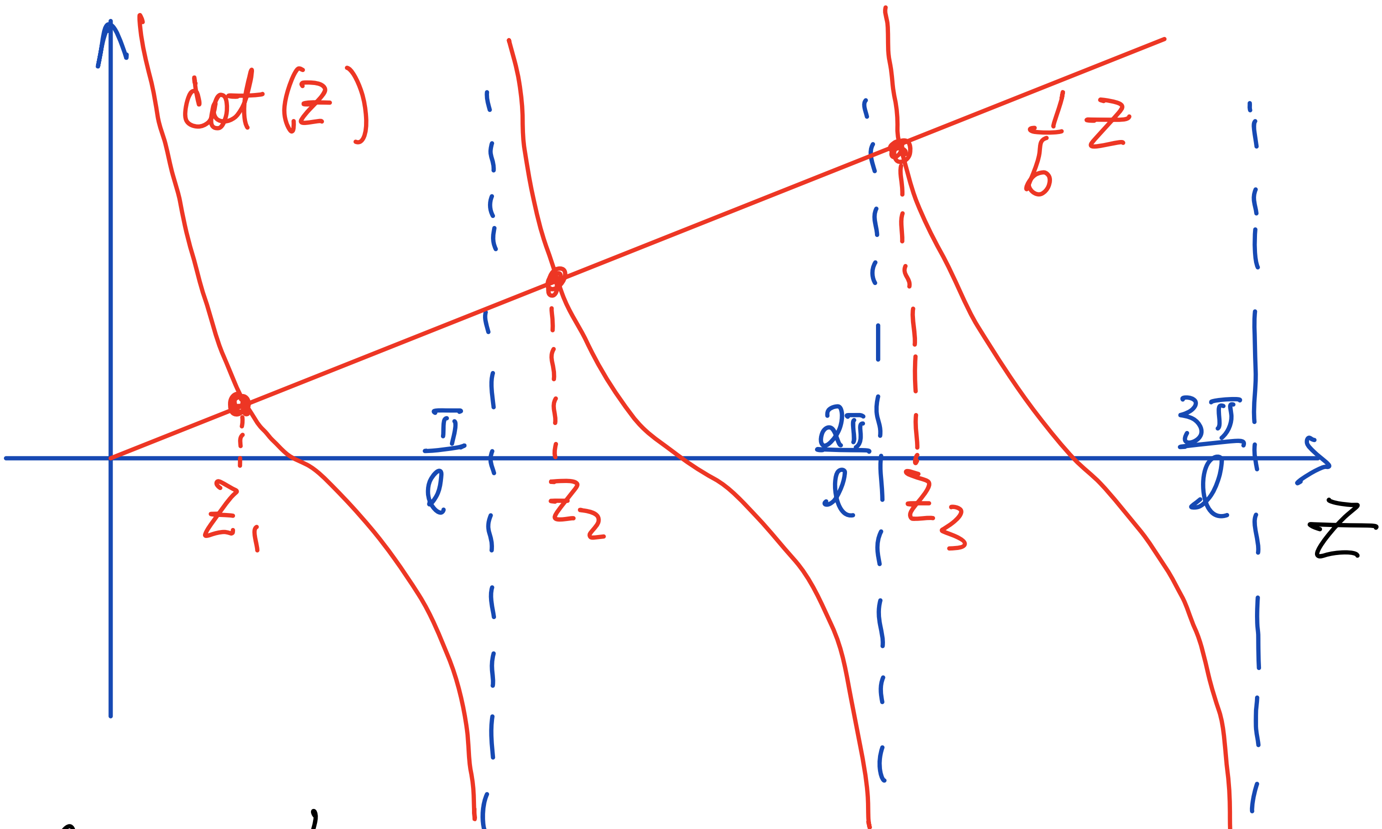
$\varphi_x(0) = 0 \Rightarrow B = 0 \Rightarrow \varphi(x) = A\cos\sqrt{-\lambda}\, x$

$\varphi_x(l) + b\varphi(l) = 0$

$$\Rightarrow -\sqrt{-\lambda}\sin(\sqrt{-\lambda}\, l) + b\cos(\sqrt{-\lambda}\, l) = 0$$

$$\Rightarrow \quad \cot(\sqrt{-\lambda}\, l) = \frac{1}{b}\sqrt{-\lambda}$$

$\cot(lz) = \frac{1}{b} z \qquad (z = \sqrt{-\lambda})$



Let $\{z_n\}_{n\geq 1}$ be the solutions of $\cot(zl) = \frac{1}{b} z$

Then $\quad \lambda_n = -z_n^2$

$$\varphi_n(x) = \cos(z_n x)$$

(2) Let $\quad u(x,t) = \sum_{n\geq 1} C_n(t) \cos(z_n x)$

$u_t = k\, u_{xx}$

$$\Rightarrow \sum_n \dot{C}_n \cos(z_n x) = \sum_n -z_n^2 C_n \cos(z_n x)$$

$$\Rightarrow \boxed{\dot{C}_n = -z_n C_n} \quad \cdots\cdots$$

Hw 6

(4.3) #1

Eq. (4.20) $$b_n''(t) + \frac{n^2\pi^2c^2}{l^2} b_n(t) = \beta_n(t)$$

Eq. (4.21) $$b_n(t) = \frac{l}{n\pi c}\int_0^t \left(\sin\frac{n\pi c(t-s)}{l}\right)\beta_n(s)\,ds$$

$$\boxed{b_n(0) = 0}$$

$$b_n'(t) = \frac{l}{n\pi c}\left(\sin\frac{n\pi c(t-s)}{l}\right)\beta_n(s)\Big|_{s=t}$$

$$+ \frac{l}{n\pi c}\int_0^t \left(\frac{n\pi c}{l}\right)\left(\cos\frac{n\pi c}{l}(t-s)\right)\beta_n(s)\,ds$$

$$= \int_0^t \left(\cos\frac{n\pi c}{l}(t-s)\right)\beta_n(s)\,ds$$

$$\boxed{b_n'(0) = 0}$$

$$b_n''(t) = \left(\cos \frac{n\pi c}{l}(t-s)\right)\beta_n(s)\Big|_{s=t}$$

$$- \frac{n\pi c}{l}\int_0^t \sin\left(\frac{n\pi c}{l}(t-s)\right)\beta_n(s)\,ds$$

$$= \beta_n(t) - \frac{n\pi c}{l}\int_0^t \sin\left(\frac{n\pi c}{l}(t-s)\right)\beta_n(s)\,ds$$

Hence $\boxed{b_n''(t) + \frac{n^2\pi^2c^2}{l^2}\, b_n(t) = \beta_n(t)}$

(4.3) #2

$$\begin{cases} u_{tt} = c^2 u_{xx} & 0 < x < l \\ u_x(0,t) = 0, \quad u_x(l,t) = 0 \\ u(x,0) = bx \\ u_t(x,0) = 0 \end{cases}$$

$$\varphi_n(x) = \cos\left(\frac{n\pi x}{l}\right), \quad n = 0, 1, 2, \ldots$$

$$\lambda_n = -\frac{n^2\pi^2}{l^2}$$

$$u(x,t) = \sum_{n \geq 0} C_n(t)\varphi_n(x)$$

$$\sum_{n \geq 0} \ddot{C}_n(t)\varphi_n(x) = \sum_{n \geq 0} c^2 \lambda_n C_n(t)\varphi_n(x)$$

$$\ddot{C}_n(t) = c^2 \lambda_n C_n(t)$$

$$C_n(t) = A_n \cos\frac{cn\pi}{l}t + B_n \sin\frac{cn\pi}{l}t$$

$$\dot{C}_n(t) = -A_n\frac{cn\pi}{l}\sin\frac{cn\pi t}{l} + B_n\frac{cn\pi}{l}\cos\frac{cn\pi t}{l}$$

$$\dot{C}_n(0) = 0 \implies B_n = 0$$

$$C_n(t) = A_n \cos\left(\frac{cn\pi}{\ell}t\right)$$

Hence $u(x,t) = \sum_{n\geq 0} A_n \cos\left(\frac{cn\pi t}{\ell}\right)\cos\left(\frac{n\pi x}{\ell}\right)$

$$u(x,0) = \sum_{n\geq 0} A_n \cos\left(\frac{n\pi x}{l}\right) = bx$$

$$A_0\, \ell = b\int_0^\ell x\, dx = \frac{b\ell^2}{2}, \qquad A_0 = \frac{b\ell}{2}$$

$$\underbrace{A_n \int_0^\ell \cos^2\left(\frac{n\pi x}{\ell}\right) dx} = \underbrace{\int_0^\ell bx \cos\left(\frac{n\pi x}{\ell}\right) dx}$$

$$= \frac{\ell}{\pi}\int_0^\pi \cos^2(nx)\, dx \qquad = \frac{b\ell^2}{\pi^2}\int_0^\pi x\cos nx\, dx$$

$$= \frac{\ell}{2\pi}\pi \qquad = \frac{b\ell^2}{\pi^2}\left[\cancel{\frac{x\sin nx}{n}\Big|_0^\pi} - \frac{1}{n}\int_0^\pi \sin nx\, dx\right]$$

$$= \frac{\ell}{2} \qquad = \frac{b\ell^2}{\pi^2}\frac{1}{n^2}(\cos nx)\Big|_0^\pi = \frac{b\ell^2}{\pi^2 n^2}\left((-1)^n - 1\right)$$

$$A_n = \frac{2b\ell}{\pi^2 n^2}\left((-1)^n - 1\right)$$

$$\boxed{\begin{aligned} &u(x,t) \\ &= \frac{bl}{2} + \sum_{n\geq 1} \frac{2bl}{\pi^2 n^2}\left((-1)^n - 1\right)\cos\left(\frac{cn\pi t}{l}\right)\cos\left(\frac{n\pi x}{l}\right) \end{aligned}}$$

(b) $u_t(x,t)\Big|_{x=0}$

$$= \sum_{n\geq 1} \frac{2b\cancel{l}}{\pi^{\cancel{2}} n^{\cancel{2}}}\left((-1)^n - 1\right)\frac{c\cancel{n\pi}}{\cancel{l}}\sin\left(\frac{cn\pi t}{l}\right)$$

$\neq 0$ only if n is odd

$$= \frac{2bc}{\pi}\sum_{n\geq 1} \frac{\left((-1)^n - 1\right)}{n}\sin\left(\frac{cn\pi t}{l}\right)$$

$$= \frac{2bc}{\pi}\sum_{n\geq 1} \frac{(-2)}{(2n-1)}\sin\left(\left(\frac{c\pi}{l}\right)(2n-1)t\right)$$

$$= \underbrace{(-bc)}_{|bc|}\underbrace{\left[\frac{4}{\pi}\sum_{n\geq 1}\frac{1}{2n-1}\sin\left((2n-1)\left(\frac{c\pi t}{l}\right)\right)\right]}_{\pm 1 \quad \text{Table 2.1 \#6}}$$

(4.3) #5

$$u_{tt} = c^2 u_{xx} - a^2 u \qquad 0 < x < l$$

$$u(0,t) = u(l,t) = 0$$

$$\varphi_n(x) = \sin\left(\frac{n\pi x}{l}\right), \quad \lambda_n = -\frac{n^2\pi^2}{l^2}$$

$$u(x,t) = \sum_{n \geq 1} C_n(t)\, \varphi_n(x)$$

$$\sum_n \ddot{C}_n(t)\varphi_n(x) = \sum_n c\lambda_n C_n \varphi_n - \sum_n a^2 C_n \varphi_n$$

$$\ddot{C}_n(t) = (c\lambda_n - a^2)\, C_n$$

$$= -\left(\frac{c n^2\pi^2}{l^2} + a^2\right) C_n$$

$$\boxed{C_n(t) = A_n \cos\left(\sqrt{\frac{c n^2\pi^2}{l^2} + a^2}\; t\right) + B_n \sin\left(\sqrt{\frac{c n^2\pi^2}{l^2} + a^2}\; t\right)}$$

(4.3) #8 $\quad u_{tt} = c^2 u_{xx} \qquad 0 < x < \ell$

M1 $\quad u(0,t) = u(\ell,t) = 0$

$$\varphi_n(x) = \sin\frac{n\pi x}{\ell}, \quad \lambda_n = -\frac{n^2\pi^2}{\ell^2}, \quad n \geq 1$$

$$u(x,t) = \sum_{n\geq 1}\left(A_n \cos\frac{cn\pi t}{\ell} + B_n \sin\frac{cn\pi t}{\ell}\right)\sin\frac{n\pi x}{\ell}$$

$$u_x = \sum_{n\geq 1}\underbrace{\left(\frac{n\pi}{\ell}\right)\left(A_n \cos\frac{cn\pi t}{\ell} + B_n \sin\frac{cn\pi t}{\ell}\right)}_{M_n}\cos\frac{n\pi x}{\ell}$$

$$u_t = \sum_{n\geq 1}\underbrace{\left(\frac{cn\pi}{\ell}\right)\left(-A_n \sin\frac{cn\pi t}{\ell} + B_n \cos\frac{cn\pi t}{\ell}\right)}_{N_n}\sin\frac{n\pi x}{\ell}$$

By Parseval's Identity,

$$\int_0^\ell u_t^2\,dx = \sum_{n\geq 1} N_n^2 \cancel{\left\|\sin\frac{n\pi x}{\ell}\right\|_{L^2}^2} \; \frac{\ell}{2}$$

$$= \sum_{n\geq 1} \frac{c^2 n^2 \pi^2}{\ell^2} \frac{\ell}{2} \left(-A_n \sin \frac{cn\pi t}{\ell} + B_n \cos \frac{cn\pi t}{\ell}\right)^2$$

$$\underline{\int_0^\ell c^2 u_x^2 \, dx} = \sum_{n\geq 1} c^2 M_n^2 \left\| \cancel{\cos \frac{n\pi x}{\ell}} \right\|_{L^2}^2 \; \frac{\ell}{2}$$

$$= \sum_{n\geq 1} \frac{c^2 n^2 \pi^2}{\ell^2} \frac{\ell}{2} \left(A_n \cos \frac{cn\pi t}{\ell} + B_n \sin \frac{cn\pi t}{\ell}\right)^2$$

$$+ \Longrightarrow \boxed{\sum_{n\geq 1} \frac{c^2 n^2 \pi^2}{2\ell} \left(A_n^2 + B_n^2\right)}$$

<u>Does not depend on t</u>

(M2) Let $E(t) = \int_0^\ell \left(u_t^2 + c^2 u_x^2\right) dx$

$$\frac{d}{dt} E(t) = 2 \int_0^\ell \left(u_t u_{tt} + c^2 u_x u_{xt}\right) dx$$

$$= 2 \int_0^\ell u_t u_{tt} \, dx + 2c^2 \int_0^\ell u_x (u_t)_x \, dx$$

$$= 2\int_0^l u_t u_{tt}\,dx + 2c^2 \underbrace{\cancel{u_x u_t\Big|_0^l}}_{0} - 2c^2\int_0^l u_{xx} u_t\,dx$$

$u(0,t)=0 \Rightarrow u_t(0,t)=0$

$u(l,t)=0 \Rightarrow u_t(l,t)=0$

$$= 2\int_0^l u_t \underbrace{\cancel{(u_{tt} - c^2 u_{xx})}}_{0}\,dx$$

$$= 0$$

Hence $\frac{d}{dt}E(t) = 0$

i.e. $E(t) \equiv E(0)$, does not depend on t.

(4.4) #1

$$u(x,y) = \sum_{n\geq 1}\left(\sin\frac{n\pi x}{\ell}\right)\left(A_n e^{\frac{n\pi y}{\ell}} + B_n e^{-\frac{n\pi y}{\ell}}\right)$$

$(u(0,y) = u(\ell,y) = 0)$ $\quad A_n + B_n = 0 \Longleftrightarrow u(x,0) = 0$

$$x(\ell - x) = \sum_{n\geq 1} \sin\left(\frac{n\pi x}{\ell}\right)\left(A_n e^{n\pi} + B_n e^{-n\pi}\right)$$

$B_n = -A_n$

(4.4) #2

$n = 0$

$$u(x,y) = A_0 + B_0 y + \sum_{n\geq 1}\left(\cos\frac{n\pi x}{\ell}\right)\left(A_n e^{\frac{n\pi y}{\ell}} + B_n e^{-\frac{n\pi y}{\ell}}\right)$$

$u(x,0) = 0$

$u_x(0,y) = u_x(\ell,y) = 0 \quad \Longrightarrow \quad A_0 = 0, \quad A_n = -B_n$

$$\underbrace{u(x,\ell)}_{=\,x} = B_0 \ell + \sum_{n\geq 1}\left(\cos\frac{n\pi x}{\ell}\right)\left(A_n e^{n\pi} + B_n e^{-n\pi}\right)$$

(4.4) #3

$$u(x,y) = A_0 + B_0 y + \sum_{n \geq 1} \left(\cos \frac{n\pi x}{l}\right)\left(A_n e^{\frac{n\pi y}{l}} + B_n e^{-\frac{n\pi y}{l}}\right)$$

$$u_x(0,y) = u_x(l,y) = 0$$

$$u_y(x,y) = B_0 + \sum_{n \geq 1} \left(\cos \frac{n\pi x}{l}\right)\left(\frac{n\pi}{l}\right)\left(A_n e^{\frac{n\pi y}{l}} - B_n e^{-\frac{n\pi y}{l}}\right)$$

$$u_y(x,0) = 0 \Longrightarrow B_0 = 0 \;\&\; A_n = -B_n$$

$$u_y(x,l) = \sum_{n \geq 1} \left(\cos \frac{n\pi x}{l}\right)\left(\frac{n\pi}{l}\right)\underbrace{\left(A_n e^{n\pi} - B_n e^{-n\pi}\right)}$$

$$f(x)$$

$$\int_0^l \cos \frac{n\pi x}{l}\, dx = 0 \text{ for any } n \Rightarrow \int_0^l f(x)\,dx = 0$$

(4.4) #4

$$u(x,y) = \sum_n \left(\sin\frac{n\pi x}{\ell}\right)\left(\cancel{A_n e^{\frac{n\pi y}{\ell}}} + B_n e^{-\frac{n\pi y}{\ell}}\right)$$

0 so that it remains bounded as $y \to +\infty$

$$u(x,y) = \sum_n \left(\sin\frac{n\pi x}{\ell}\right)\left(B_n e^{-\frac{n\pi y}{\ell}}\right)$$

$$f(x) = \sum_{n\geq 1} B_n \sin\frac{n\pi x}{\ell}$$

Hw 7

(4.4) #5

$$u(r,\theta) = A_0 + B_0 \log r + \sum_{n\geq 1} \left(A_n r^n + B_n r^{-n}\right)\cos n\theta + \sum_{n\geq 1}\left(C_n r^n + D_n r^{-n}\right)\sin n\theta$$

$$\partial_r u(r,\theta) = \frac{B_0}{r} + \sum_{n\geq 1}\left(nA_n r^{n-1} - nB_n r^{-n-1}\right)\cos n\theta + \sum_{n\geq 1}\left(nC_n r^{n-1} - nD_n r^{-n-1}\right)\sin n\theta$$

$r = r_0$

$$0 = \frac{B_0}{r_0} + \sum_{n\geq 1}\left(nA_n r_0^{n-1} - nB_n r_0^{-n-1}\right)\cos n\theta + \sum_{n\geq 1}\left(nC_n r_0^{n-1} - nD_n r_0^{-n-1}\right)\sin n\theta$$

$$\Longrightarrow \boxed{B_0 = 0, \quad \begin{array}{l} A_n r_0^{n-1} = B_n r_0^{-n-1} \\ C_n r_0^{n-1} = D_n r_0^{-n-1} \end{array}}$$

$r = 1$

$$\boxed{f(\theta) = A_0 + \sum_{n\geq 1}(A_n + B_n)\cos n\theta + (C_n + D_n)\sin n\theta}$$

(4.4) #6

$$u(r,\theta) = \frac{A_0}{2} + \sum_{n\geq 1}\left(A_n r^n \cos n\theta + B_n r^n \sin n\theta\right)$$

(a) $$u(R,\theta) = \frac{A_0}{2} + \sum_{n\geq 1}\left(A_n R^n \cos n\theta + B_n R^n \sin n\theta\right)$$

$$\frac{1}{2\pi}\int_0^{2\pi} u(R,\theta)\,d\theta = \frac{A_0}{2} + (0)$$

$$||$$

$$u(0,\theta) = \frac{A_0}{2}$$

(b) $$P(r,\theta) = \frac{1-r^2}{r^2 - 2r\cos\theta + 1} \qquad \begin{array}{l} 0 \leq r < 1 \\ 0 \leq \theta < 2\pi \end{array}$$

$$= \underbrace{\frac{1}{2\pi}\left[1 + 2\sum_{n\geq 1} r^n (\cos n\theta)\right]}$$

$1 - r^2 > 0$

$r^2 - 2r\cos\theta + 1 > 0$

$$\left(\begin{array}{l}\Delta = b^2 - 4ac \\ \quad = 4\cos^2\theta - 4 < 0\end{array}\right)$$

$$\int_0^{2\pi} (\qquad)\, d\theta = 1$$

(c) $$u(r,\theta) = \int_0^{2\pi} \underline{f(\psi)} P(r, \psi-\theta)\, d\psi \qquad |f| \leq M$$

$$|u(r,\theta)| \leq \underline{M}\int_0^{2\pi} |P|\, d\psi = M\int_0^{2\pi} P\, d\psi = M$$

(4.4) #7. Refer to posted Note of Week 9

Hw 8

(5.1) #3

$$Y_\nu(x) = \frac{\cos(\nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}, \quad \nu > 0, \neq n$$

$$= \frac{\cos(\nu\pi)\left[\frac{x^\nu}{2^\nu \Gamma(\nu+1)} + \cdots\right] - \frac{x^{-\nu}}{2^{-\nu}\Gamma(-\nu+1)}}{\sin(\nu\pi)}$$

dominating term as $x \to 0^+$

$$\cong -\frac{x^{-\nu}}{2^{-\nu}(\sin\nu\pi)\Gamma(-\nu+1)}$$

$$\Gamma(\nu)\Gamma(-\nu+1) = \frac{\pi}{\sin\nu\pi}$$

$$= -\frac{x^{-\nu}}{2^{-\nu}\,\pi/\Gamma(\nu)}$$

$$= -\frac{\Gamma(\nu)\, x^{-\nu}}{2^{-\nu}\pi} \xrightarrow{\nu \to n} -\frac{(n-1)!}{\pi}\left(\frac{x}{2}\right)^{-n}$$

$\lim_{\nu \to n}$ ✓

(5.1) #4

$$Y_\nu(x) = \frac{\cos(\nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

$\nu \to n$

L'H Rule

$$Y_n(x) = \left. \frac{-\pi(\sin\nu\pi) J_\nu(x) + (\cos\nu\pi)\partial_\nu J_\nu(x) - \partial_\nu(J_{-\nu}(x))}{\pi\cos(\nu\pi)} \right|_{\nu=n}$$

$$= \frac{1}{\pi}\left[\partial_\nu J_\nu(x) + (-1)^{n+1}\partial_\nu\left(J_{-\nu}(x)\right)\right]\Bigg|_{\nu=n}$$

$\cos(n\pi) = (-1)^n$

(5.1) #5

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}$$

$$(\partial_\nu J_\nu)(x) = \sum_{k=0}^{\infty} \frac{-(-1)^k}{k!\,\Gamma(k+\nu+1)^2}\,\Gamma'(k+\nu+1)\left(\frac{x}{2}\right)^{2k+\nu} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}\log\left(\frac{x}{2}\right)$$

Hence

$$(\partial_\nu J_\nu)(x)\Big|_{\nu=0} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!\,\Gamma(k+1)}\,\psi(k+1)\left(\frac{x}{2}\right)^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+1)}\left(\frac{x}{2}\right)^{2k}\log\left(\frac{x}{2}\right)$$

$J_0(x)$

$$= \left(\log\frac{x}{2}\right)\sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}\psi(k+1)}{(k!)^2}\left(\frac{x}{2}\right)^{2k}$$

$$= J_0(x)\log\frac{x}{2} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}\psi(k+1)}{(k!)^2}\left(\frac{x}{2}\right)^{2k}$$

Note:

$$\partial_\nu\left(J_{-\nu}(x)\right) = \left(\partial_\nu J\right)_{(-\nu)} \frac{\partial(-\nu)}{\partial\nu}\Big|_{\nu=0} = -\left(\partial_\nu J\right)_{\nu=0}$$

Hence

$$\underline{Y_0(x)}$$

$$= \frac{1}{\pi}\left[\partial_\nu J_\nu\Big|_{\nu=0}(x) + \cancel{(-1)^{0+1}}\,\cancel{(-1)}\left(\partial_\nu J\right)_{\nu=0}\right]$$

$$= \frac{2}{\pi}\,\partial_\nu J_\nu\Big|_{\nu=0}(x)$$

$$= \boxed{\frac{2}{\pi}\left[J_0(x)\log\frac{x}{2} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}\psi(k+1)}{(k!)^2}\left(\frac{x}{2}\right)^{2k}\right]}$$

(5.2) #1

$$J_{\frac{1}{2}}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\frac{3}{2})} \left(\frac{x}{2}\right)^{2k+\frac{1}{2}}$$

$$= \left(\frac{x}{2}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\frac{3}{2})} \left(\frac{x}{2}\right)^{2k}$$

$$= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{x}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2(4)\cdots(2k)\,\Gamma(k+\frac{3}{2})} \frac{x^{2k+1}}{2^k}$$

$$\Gamma\left(k+\frac{3}{2}\right) = \left(k+\frac{1}{2}\right)\left(k-1+\frac{1}{2}\right)\cdots\left(\frac{1}{2}+1\right)\Gamma\left(\frac{1}{2}+1\right)$$

$$= \left(k+\frac{1}{2}\right)\left(k-1+\frac{1}{2}\right)\cdots\left(\frac{1}{2}+1\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)$$

$$\left(\text{Hence } 2^k\,\Gamma\left(k+\frac{3}{2}\right) = \sqrt{\pi}\,(1)(3)(5)\cdots(2k+1)\frac{1}{2}\right)$$

$$= \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{2k+1}}{(2k+1)!} = \sqrt{\frac{2}{\pi x}} \sin x$$

(5.2) #2 Use (5.17) with $\nu = -\frac{1}{2}$

$$x J_{-\frac{3}{2}}(x) + x J_{\frac{1}{2}}(x) = 2\left(-\frac{1}{2}\right) J_{-\frac{1}{2}}(x)$$

$$J_{-\frac{3}{2}}(x) = -J_{\frac{1}{2}}(x) - \frac{1}{x} J_{-\frac{1}{2}}(x)$$

$$= -\sqrt{\frac{2}{\pi x}}\left(\sin x + \frac{1}{x}\cos x\right)$$

Also, $\nu = \frac{1}{2}$

$$x J_{-\frac{1}{2}}(x) + x J_{\frac{3}{2}}(x) = 2\left(\frac{1}{2}\right) J_{\frac{1}{2}}(x)$$

$$J_{\frac{3}{2}}(x) = -J_{-\frac{1}{2}}(x) + \frac{1}{x} J_{\frac{1}{2}}(x)$$

$$= \sqrt{\frac{2}{\pi x}}\left[-\cos x + \frac{1}{x}\sin x\right]$$

(5.3) #2

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

$$\left(J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\,\overbrace{\cos\left(x+\frac{\pi}{4}-\frac{\pi}{4}\right)}^{\equiv} + \overbrace{\cancel{O\left(\frac{1}{x^{3/2}}\right)}}^{\equiv 0}\right)$$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$

$$\left(J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\,\overbrace{\cos\left(x-\frac{\pi}{4}-\frac{\pi}{4}\right)}^{\equiv} + \overbrace{\cancel{O\left(\frac{1}{x^{3/2}}\right)}}^{\equiv 0}\right)$$

$$J_{-\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(-\sin x\right) - \underbrace{\sqrt{\frac{2}{\pi}}\,\frac{1}{x^{3/2}}\cos x}_{\equiv}$$

$$\left(J_{-\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\,\underbrace{\cos\left(x+\frac{3\pi}{4}-\frac{\pi}{4}\right)}_{\equiv -\sin x} + \underbrace{O\left(\frac{1}{x^{3/2}}\right)}\right)$$

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(-\cos x\right) + \sqrt{\frac{2}{\pi}}\frac{1}{x^{3/2}}\sin x$$

$$\|\qquad\qquad\qquad\qquad\qquad\qquad\|$$

$$\left(J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi}}\cos\left(x - \frac{3}{4}\pi - \frac{\pi}{4}\right) + O\left(\frac{1}{x^{3/2}}\right)\right)$$

$$\cos(x-\pi) = -\cos x$$