MA 520 Spring 2024 (Aaron N. K. Yip)
Homework 4
Due: Thursday, Feb. 1st, in class

Folland: Fourier Analysis and Its Applications
Section 2.4 (p.47): #1, 2, 3, 4; (For each of the problems, plot also the versions of the function $f(\theta)$ on $(-\pi, \pi)$ which you use to compute the sin and cos series.)
Section 3.1 (p.67): #1, 2, 3, 4, 5.

Additional Problem
In class, we have computed the N-th partial Fourier series $(S_N f)$ of a 2π-periodic function f and express it as a convolution with the Dirichlet kernel, $D_N(\cdot)$:

$$(S_N f)(x) := \sum_{n=-N}^{N} c_n e^{inx} = \int_{-\pi}^{\pi} f(y) D_N(x-y)\, dy$$

where D_N is defined by and computed as ([F, p.33-34, (2.12), (2.14)),

$$D_N(\theta) := \frac{1}{2\pi}\sum_{n=-N}^{N} e^{in\theta} = \frac{\sin\left(\left(N+\frac{1}{2}\right)\theta\right)}{2\pi\sin(\frac{1}{2}\theta)}.$$

Recall the following properties of D_N:

1. $D_N(\cdot)$ is 2π-periodic;
2. $D_N(\cdot)$ is an even function;
3. $\displaystyle\int_{-\pi}^{\pi} D_N(\theta)\, d\theta = 1$;
4. $D_N(0)\ (= \lim_{\theta\to 0} D_N(\theta)) \ = \dfrac{2N+1}{2\pi}$;
5. $D_N(\theta) \sim \dfrac{\sin((N+\frac{1}{2})\theta)}{2\pi}$ for $\theta \sim \pm\pi$. Note that the *magnitude of the oscillations* of D_N for θ near π remains $\frac{1}{2\pi}$ which *does not diminish* for $N \gg 1$.

The graph of D_N is given in [F, p.34, Fig. 2.4].

Now consider the "average" of the partial sums:

$$(A_N f)(x) := \frac{1}{N+1}\sum_{m=0}^{N}(S_m f)(x) = \frac{(S_0 f)(x) + (S_1 f)(x) + \cdots + (S_N f)(x)}{N+1}$$

(In the above, we define $(S_0 f) = \frac{a_0}{2}$, the first term of the Fourier series of f, i.e. $(S_0 f)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, dy$.) Folland has some discussion of this notion – also called the Cesàro mean – in p.60.

1. Show that $A_N f$ can also be given by a convolution of f but with a new kernel F_N, called the *Fejér* kernel:

$$(A_N f)(x) = \int_{-\pi}^{\pi} f(y) F_N(x-y)\, dy.$$

Find the explicit formula for $F_N(\theta)$.

Hint: Note that $F_N(\theta) = \frac{1}{N+1}\big(D_0(\theta) + D_1(\theta) + \cdots + D_N(\theta)\big)$. One way to compute this sum is to replace $\sin\theta$ by $\frac{e^{i\theta}-e^{-i\theta}}{2i}$ and use the formula for the summation of geometric series.

2. For F_N, discuss the five previously stated properties of D_N, i.e. are they true for F_N? If not, give the corresponding correct statements for F_N. Pay extra attention to Property 5.

3. Show that for any $\epsilon > 0$ and $\delta > 0$, there exists an $N_*(\epsilon,\delta) > 0$ such that for any $|\theta| > \delta$ and $N \geq N_*(\epsilon,\delta)$, it holds that

$$|F_N(\theta)| \leq \epsilon, \quad \text{for any} \quad |\theta| \geq \delta.$$

In words, "as long as θ stays away from 0, the Fejèr Kernel F_N can be made as small as possible if N is large enough".

Find $N_*(\epsilon,\delta)$ for $\epsilon = 10^{-6}$ and $\delta = 10^{-5}$.

4. Use (any of your favorite computer software to) draw $D_{25}(\theta)$ and $F_{25}(\theta)$ for $-\pi < \theta\pi$ (similar to that for D_N (see [F, p.34, Fig. 2.4]).

Extra information.

Note that convergence of $A_N f$ to f is much better behaved than that of $S_N f$. In particular, there is *no* Gibbs phenomena for $A_N f$. See the following figures for the actual comparison between $S_N f$ and $A_N f$ for the square function.