MA 520 Spring 2024 (Aaron N. K. Yip)

Homework 9

Due: Sunday, Apr. 21st, 11:59pm, submit directly in Gradescope

Folland:

Section 6.2: #6, 7;

Section 6.3: #3 (describe also the "dominating behavior" of u as $r \longrightarrow \infty$), 6, 8.

Section 6.4: #4, 5;

Section 6.5: #5, 6;

Additional Problem.

1. Show *explicitly* that the first three Hermite polynomials, H_0, H_1, H_2, are orthgonal in $L^2_\omega(\mathbf{R})$ with weight $\omega(x) = e^{-x^2}$.

   (Of course, we know that they are orthogonal, by [F. Theorem 6.11].)

2. Show *explicitly* that the first three Laguerre polynomials, $L^\alpha_0(x), L^\alpha_1(x), L^\alpha_2(x)$ (for $\alpha > -1$) are orthogonal in the space $L^2_\omega(0,\infty)$ with weight $\omega(x) = x^\alpha e^{-x}$.

   (Of course, we know that they are orthogonal, by [F. Theorem 6.15].)

3. (Related to wave function of the hydrogen atom.)

   Consider a radially symmetric function $v(x) = v(r)$ that solves:

   $$v_{rr} + \frac{a}{r}v_r + \frac{b}{r}v = \lambda v$$

   where $\lambda > 0$ and a and b are some constants.

   (a) Define $w(r) = e^{\beta r}v(r)$ with $\beta = \sqrt{\lambda}$. Find the equation satisfied by w.

   (b) Define $y(x) = w(r)$ where $x = cr$, i.e. $y(cr) = w(r)$. Find the equation satisfied by $y(x)$.

   (c) Find the values of λ and c (in terms of a and b) such that y can be given by a Laguerre polynomial $L^\alpha_n(x)$.

   (Hint: write down the equation of $y(x)$ and compare it with [F. p. 192, (6.47)].)

   (d) Write down the final formula for the function v in terms of Laguerre polynomials L^α_n.