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FOURIER'S SERIES

The Genesis and Evolution of a Theory

By

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Professor of Mathematics

in the

University of Wisconsin

The First

HERBERT ELLSWORTH SLAUGHT

MEMORIAL PAPER

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PART II

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R. E. LANGER

Number 1
of the
HERBERT ELLSWORTH SLAUGHT
MEMORIAL PAPERS

AUGUST-SEPTEMBER 1947

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THE HERBERT ELLSWORTH SLAUGHT MEMORIAL PAPERS

The editorial committee is happy to present this, the first of the Slaught Memorial Papers. In view of the very great interest which Professor R. E. Langer has had in these Papers from their very inception, it is particularly fitting that he should be the author of the first of the series.

At this time it may be appropriate to give a brief account of the origin of these Papers, and an indication of their purpose. This can perhaps best be done by quoting from the reports of two different committees of the Association. In the MONTHLY for February, 1940, there appears the report of a "committee to review the activities of the Mathematical Association of America." Along with other recommendations, this report, which was prepared by Professor Langer as chairman, suggested that the Association establish a series of expository pamphlets which might well take the form of a memorial to Professor Slaught. Subsequently, another committee was appointed, with Professor C. V. Newsom as chairman, to make specific recommendations for the establishment of such a series. The report of this committee appeared in the MONTHLY for February, 1941.

From the report of the "Langer Committee" we may quote as follows:

"The encouragement and sponsorship of expository and critical writing is one of the objectives of the Association which enjoys the unanimous support of the members. There is a ready welcome and a general demand for more readable scholarly papers on all kinds of mathematical subjects from the classical to the modern, from the elementary to the advanced, on theory, on applications, on history, or on philosophy. In the past there have, of course, been the Carus monographs, and from time to time excellent papers in the MONTHLY. There seems, however, to be at the present little or no means for the ready publication of writings which in length are intermediate betwen the relatively few pages of a journal paper, and the relatively many pages of a complete monograph. Such papers, say in length between twenty and a hundred pages, could be profitably written on subjects in many categories, including among others, elementary introductory expositions of theories and their applications, more advanced expositions and interpretations of modern viewpoints and theories, philosophical essays and criticisms, broad historical accounts of important schools, or biographical accounts of individuals."

The report of the "Newsom Committee" said in part:

"The principal conclusion reached by this investigation is that there is a widespread interest in additional expository writing of the type discussed in the report of the 'Langer Committee,' and that those who are now sponsoring series of expository monographs would welcome the creation of additional opportunities for the publication of studies pertaining to mathematical subjects. In truth, the members of the committee have been impressed with the enthusiasm which has been displayed by those who have given opinions relative to the possibility of a new publication program sponsored by the Association. Syntheses of modern investigations in many fields of mathematics seem to be wanted by college men who do not have an opportunity to follow developments in the mathematical literature. Instructors in our junior colleges and secondary schools who may have a limited preparation in mathematics are seeking easily accessible accounts of some of the older theories. Some correspondents have expressed the belief that there is an amazing dearth of readable mathematical material for college students who have studied little beyond the calculus. And finally, some have emphasized that the interest of the American public in mathematical attainments and methods needs to be cultivated; this interest is attested to by the recent wide sale of a few popular books upon mathematics."

In accordance with the recommendations of the "Newsom Committee," the Board of Governors authorized a series of expository pamphlets to be known as the "Herbert Ellsworth Slaught Memorial Papers." The long delay in the actual appearance of the first of these Papers was largely caused by the demands of the war which left little or no time for the writing of mathematical expositions.

The Slaught Memorial Papers are to be published in the form of supplements to the MONTHLY and, at least for the present, are being sent free to all subscribers. The success of this project will depend on the interest of mathematicians generally and, more particularly, upon the co-operation of competent scholars who will be willing to devote sufficient effort to the difficult but worth-while task of writing elementary expositions of their respective fields of interest.

The editorial committee through the undersigned will welcome suggestions from any interested persons and, in particular, will be glad to hear from prospective authors of expository articles which might be suitable for publication in this series.

N. H. McCoy

FOURIER'S SERIES

THE GENESIS AND EVOLUTION OF A THEORY

R. E. LANGER

PREFACE

In choosing to present in this exposition some chapters from the theory of the representation of arbitrary functions in infinite series, I have done so in the belief that this subject has an unusually broad appeal. For in singular measure it serves both theoretical and practical ends. The pure analyst finds in it a wealth of structure and subtle inter-relationship, while the applied mathematician and the related scientist find in it, no less, a tool of almost endless flexibility and use.

The simpler formal elements of the theory of trigonometrical infinite series are, it may be assumed, in some measure familiar to all who aspire to a level of mathematical attainment above the elementary one. Presentations of them in text-book form are common, and many of them eminently readable. It is not my purpose to duplicate any such expositions of fact and procedure, but rather to present here other matters less usually considered. These I have, in the main, centered about two focal theses, namely first, a sort of case history of the inception of the theory and its development to the stage attained by Fourier, and second, a generalization of the theory in which the trigonometrical form of it is subordinated to the status of a mere special case.

In its modern form the theory of Fourier's series and its applications to problems of physics admit of presentation in a direct and logical manner that is, on the whole, strikingly economical in design. The reasoning is straight-forward and to the point, and has at almost every turn an aspect of complete inevitability. The trigonometric formulas invariably appear to fit the needs at issue with such precision and neatness as could not have been more so had they been specifically tailored to the purpose. So completely is this true, that it seems no far cry to the suggestion that the whole structure might be the creation of some single master architect, who, in his genius, could draw to hand the exact and unerring means for an orderly and consummate unfolding of all the whole essential machinery of thought and analysis. Of erstwhile possible deficiencies, no trace is left revealed.

Well developed mathematical theories are prone to seem like that, and in the deceptiveness of this there is weakness as well as strength. The craftsman, whose concern with the theory is motivated by the mere search for a tool, naturally has small interest in the cruder forms of it that are now obsolete. For the student whose concern is with ideas no less than with facts, on the other hand, the too finished result is often concealing as well as revealing. The confusion of germinal ideas, the labor and stumbling of the early advance, the frustrations in imprecise notions—all these are matters which for a speedy mastery of the facts are well left aside. But precisely in these as in none other are to be discerned the creative imagination, the initial inductions and the logical strategy by which the final result was shaped. When, as in the present instance, the ingenuity and the technical exploits which gave the impetus and direction were those of such masters as the Bernoullis, Euler, d'Alembert, Lagrange and Fourier, it need hardly be feared that a review of the course of the developments will prove to be an unrewarding venture. In the growth of mathematics, much more than in its refined and polished results, is its living character evidenced.

Generalization is the medium through which the mathematician constantly seeks the enlargement of his conceptions and understanding. The vista revealed by an extant theory may be broad, but it is broadened further by generalization. And from the more expansive viewpoint the scene may be revealed not only more amply but also more distinctly. A greater simplicity in the intrinsic plan may be discernible, for many features originally judged to be quite essential may be shown, on the contrary, to be in fact merely incidental or fortuitous. This is quite the case with the Fourier theory. Its dependence upon the trigonometric formulas of combination is so conspicuous as to seem to be the very essence of it. And if in the related theories of representations in series of Bessel functions or Legendre polynomials *etc.* other combination formulas are basic, these in turn generally seem, if anything, even more specialized. A true generalization, from which the Fourier theory may be drawn forth as a special case, is the theory of ordinary differential boundary problems in which the fundamental interval of the variable is one upon which the differential equation is without singular points.

The discussion which I have given here is intended to serve these two purposes. In the first part the theme is historical. It centers about the incipience and the classical development of the theory, and is in fact a digest of some works by different masters through which conspicuous advances were made. This comes to its terminus with the discussion of Fourier's deductions, and therewith the historical thread is definitively dropped. In the second part, which is devoted to the generalization, the purpose is purely expository. The material is there set forth in as elementary a manner as I found possible, not with the generality in which it exists in the literature, but with such generality as seemed to be adequate to the display of its essential character. The fact that this larger theory embraces that of Fourier, and the manner in which it does so, is shown at appropriate points by drawing the trigonometric formulas forth as specializations obtainable without any peculiar implementations from the more general relationships derived.

I believe that in the main the paper will be readable for students who in mathematics have gone but little beyond a good course in the calculus. The simplest facts about infinite series and differential equations, the formulas for the trigonometric functions in terms of exponentials, and such, have been assumed. Beyond that all pertinent deductions have been included until the closing chapters are reached. Incidental material has, in part, been relegated to appendices. In the final chapters the elementary theory of functions of a complex variable, and in particular the theory of residues is a requisite.

In taking material from the literature, especially in the earlier parts of the paper, I have felt under no obligation to hold to the letter of the originals. The excerpts are, therefore, distinctly not to be regarded as facsimiles or verbatim reports. Although it was my intention to preserve the spirit, many formal changes were made, in part to bring the contributions from diverse sources under a consistent scheme of notation, but also in part to eliminate discursive material and to avail myself of such advantages as the presentation to modern readers might afford. The sources in the literature from which excerpts were made, or at which more extensive deductions may be found, have been indicated in the text, and are listed at the end of the paper. I have, however, made no attempt whatsoever to be complete in this matter.

Part 1
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PART I

CHAPTER 1

Introduction. It is perhaps true that few mathematical doctrines are built around facts which on first acquaintance seem so surprising as those of the infinite trigonometric series. That some such series represent functions is obvious. Simple examples are easily constructible in any abundance. How broad or how deep the adaptability of these series for functional representation may be, is, on the other hand, far from easy to see. The terms of the series are but elementary functions of a very simple type—the sines and cosines of the multiples of an angle. The theory of power series does not promise much by analogy, since those series can represent only such functions as have all the regularity which unlimited differentiability assures. It might well be expected, therefore, that the property of trigonometric representability attaches only to the functions of a quite restricted class. Historically that was the opinion which originally held sway, and which was very generally maintained. It was, in fact, so long maintained and so tenaciously, even by the greatest of mathematical masters, and in the face of most insistent evidence to the contrary, that the final breach with it took on quite definitely the character of an emancipation.

The concept of the function lies at the very heart of mathematical analysis. As it is now currently accepted it is a notion of very great breadth, covering very general interdependencies of variables upon each other. During the eighteenth century this concept was not only much more restricted, but precisely what its content and delimitations were had not yet been brought to any complete or clear formulation. Imprecise notions rarely fail to breed confusion, and in this respect the functional notion of that time was in no way an exception. It was differently conceived by different investigators. And these latter then disagreed among each other because unconsciously they talked at cross purposes. The written words flowing from different pens had different meanings. While, for instance, the function and the analytic formula were one to d'Alembert, the function was thought of as a graph by Euler, and probably meant something else again to still another.

The basis of the functional notion was originally drawn, of course, from observations upon concrete examples—in the main from such functions as we now designate as of elementary type, such as present themselves in the simpler applications of mathematics to the problems of physics. Such functions are almost invariably expressible by formulas. They generally have comparatively simple, orderly, and continuous graphs, and the identity of any two of them is restricted to isolated values of the variable. Inasmuch as this category includes no examples of distinct functions whose graphs have an entire arc in common, it was no more than natural then to consider that generally the course of a function over any interval was determinative and completely identifying, so that the graph over its entire range of definition was to be thought of as unambiguously fixed. Functional relationships such as are now commonly dealt with, in which the variables are related by different laws in different parts of an interval, were not thought of then as subsumed in any single function at all, but were regarded rather as a composite of a plurality of functional fragments. The possibility of representing such a conglomerate by a single formula was not even conceived of.

The eighteenth century stands out in mathematical history as an era of great genius. Through the work of an astonishing array of masters the science was extended and broadened by the opening of many new fields. Technical skill attained to extraordinarily high levels and new ideas were crowded one upon the other. And yet through this period the facts of the trigonometric series withheld themselves. Euler, d'Alembert, Lagrange and others walked upon the very edge of them without falling upon them. A more conspicuous example of the confining effects of preconceptions is hardly to be found. The break with all this remained to become the accomplishment of the next century, the personal achievement of Fourier. Once the step to a broader conception of the function had been made, the results of computations upon trigonometric series could be given a much more inclusive interpretation. As is now generally familiar, such series may represent functions which are not only discontinuous, but which may be quite arbitrary in the sense that over different portions of the interval they may accord with laws that need have no logical relation with each other. Computations upon even a few of the initial terms of the series often reveal these facts quite clearly.

As Fourier announced his famous theorem it was to the following effect:

Any single-valued function $f(x)$ defined over an interval $-l<x<l$, is representable over this interval by a series of sines and cosines in the manner

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[a_k \cos\frac{k\pi x}{l} + b_k \sin\frac{k\pi x}{l}\right]. \tag{1.1}$$

In this representaion the coefficients are those which are computable from the function $f(x)$ by the formulas

$$\begin{aligned} a_k &= \frac{1}{l}\int_{-l}^{l} f(s)\cos\frac{k\pi s}{l}\,ds, \\ b_k &= \frac{1}{l}\int_{-l}^{l} f(s)\sin\frac{k\pi s}{l}\,ds. \end{aligned} \tag{1.2}$$

If the interval over which the representation is to maintain is only $0<x<l$, then either sines or cosines alone suffice, the series being in the one case

$$f(x) = \sum_{k=1}^{\infty} b_k \sin\frac{k\pi x}{l}, \tag{1.3}$$

with the coefficients

$$b_k = \frac{2}{l}\int_{0}^{l} f(s)\sin\frac{k\pi s}{l}\,ds, \tag{1.4}$$

and in the other case

$$(1.5) \qquad f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \frac{k\pi x}{l},$$

with the coefficients

$$(1.6) \qquad a_k = \frac{2}{l} \int_0^l f(s) \cos \frac{k\pi s}{l}\, ds.$$

This theorem, in this utter generality, is now known, of course, to be not strictly true, even with the modern notion of the term function. The restrictions which must be applied are, however, in no way gross. They are, on the contrary, so subtle that in the pursuit of them much further clarification of analytic notions was achieved. The rôle which the trigonometric series have played in the development of precise conceptions is, on these many accounts, one of unusual interest to the student of the evolution of mathematical ideas. [1]

CHAPTER 2

Of mathematical applications to physics. By the time of the year 1725, a decade after the death of Leibniz and near the close of Newton's life, familiarity with the formal processes of the calculus had become widely disseminated, and facility in the use of these new techniques had been developed to a very substantial degree. In particular their effectiveness as instruments for the treatment of problems in mechanics had been generally recognized. This science was, therefore, under assiduous study. The more immediate of its problems—in the main those centering themselves around the motions of single mass particles—had already been pretty effectively brought to their solutions. The forefront of interest had, therefore, already been pushed beyond them to matters of greater complexity, such as presented themselves in connection with the motions of bodies with several or many degrees of freedom, or even with the reactions of flexible continuous mass distributions. Problems in the vibrations of elastic bodies in particular had begun to receive attention, and in the ensuing period these were to preoccupy an increasing number of investigators. The scientific literature of the following half century is, therefore, heavily interspersed with memoirs bearing upon this field. One may readily conjecture that the problems to be found there must have seemed almost endless in their abundance and variety. They must, moreover, have exerted a very strong fascination, if for no other reason, then because of the evident suggestion that in them lay an important key to an analytic mastery over the manifestations of nature.

Any review of the activities of the time show this to have been so. Important and difficult works were produced in great variety—among them, to name a few, investigations upon the oscillations of plates in vacuum or immersed in fluids, upon rods suspended from fixed or flexible mountings, upon jointed pendulums, upon heavy dangling chains *etc. etc.* But among all these the researches upon the motions of tautly stretched elastic strings or wires were to be especially significant. Though these were perhaps of no greater mechanical importance than the discussion of many another problem, they did contemporaneously assume a quite disproportionate prominence, and they still do so in retrospect. They became, namely, a conspicuous point of impact for many divergent conceptions and opinions, and in the rôle thereby thrust upon them they became crucial to the development not only of mechanics but of mathematics over a much wider range. Inasmuch as the ideas at issue are in large measure our essential concern in this discussion, we shall have to give a considerable modicum of our attention to this problem of the string in subsequent pages.

From the very start of investigations upon them it seems to have been assumed that continuous material bodies could, for the purposes of analysis, be approximated by systems of discrete mass particles. The conception of an extensive body as composed of particles is a very natural one. Quite apart from whether this viewpoint was intended in the end to be philosophically maintained or not, it was apparently seen to suggest more strongly than any other a practicable mode of procedure. It requires no vivid imagination to picture the discrete system as merging into the continuous one as the size of the individual particles is diminished indefinitely while their number and density is correspondingly increased. If in the analysis of a finite approximating system the formulas can be so framed and dealt with that results are deducible from them without any actual specification of the number of the particles that are involved, then this number, retaining its generality, may be assumed to figure in the results in the way of a free parameter. This parameter may then logically be made the crux of limiting considerations, and through this means the physical transition from the discrete to the continuous configuration may be thought of as implemented mathematically by a passage from the finite to the infinite.

It may well be recognized that success in the execution of any such subtle program as this would be contingent both upon superior insight and a high level of technical skill on the part of the investigator. Such is the case, no less, with almost any application of mathematics to a phenomenon of nature. The actual responses of physical bodies to ponderable influences are invariably of a discouraging degree of complexity, and this is generally due much less to the influences that are primarily under scrutiny than to the many others that are inevitable and yet really incidental and effectively irrelevant. Were complete recognition to be given to all these latter, the formulation of a natural problem would, almost without exception, be quite submerged in intricacies of detail. On the physical side the fortuitous distracting features might well obscure the salient ones, and mathematically they might well throw the problem far beyond the range of possible solution.

At the very outset, therefore, it is usual and necessary to regard the physical configuration not as it actually is, but as it might be were it to be disencumbered of all but its primarily intrinsic features. The result of this is at once a simplification and an idealization. It is this which is made the subject of analysis. The idealization is, of course, a departure from the physical reality, and this fact is certainly of no secondary importance. The original problem having been replaced by another, no immediate guarantee exists, of course, that results derived for the latter have a sufficient relevancy or applicability to the former. It is evident that with too great a divergence from the original all practical purpose would thus be defeated.

This is the consideration which checks the extent to which idealizing abstractions may be made. In determining upon these a fine sense of values and a depth of understanding are the indispensable guides. Whether in any case the permissible bounds were exceeded or not, must in the end ordinarily be decided after the fact by experiment. The applicability of the mathematically deduced theory stands or falls according as at strategic points its results agree or disagree with the data of observations. The decision, either way, implies no reflection whatever upon the theory's logical soundness. Its implications bear only upon the legitimacy of the simplifications which were made in the determination of its basis, namely upon the insight with which suitability and adequacy in the initial idealizing approximations were sensed.

CHAPTER 3

The loaded string. A natural phenomenon that is universally familiar and frequently observable in our surroundings, and to which, despite this, the attention is often sharply drawn, is that of the behavior of a stretched elastic string or wire in its response to a displacement from its state of equilibrium. In many cases this response is acoustically conspicuous, ranging from the hum of the heavy structural wire in the wind to the eloquent notes of the strings of a musical instrument. And then again, not rarely, the response is visually noticeable, being sometimes marked by so curious a feature as the presence of nodal points that maintain the state of rest while the string between them is in violent agitation.

The elastic string thus almost obtrudes itself upon the notice of the experimenter, and, having drawn his attention, recommends itself in many ways. Its geometrical configuration is of the simplest sort, permitting the identification of any of its points by a single dimension. Its motions are, under many circumstances, markedly regular, and respond promptly and prominently to adjust themselves to any quantitative modifications in the length, the tension, the weight or the initial state. It can hardly be looked upon as surprising, in virtue of all this, that the string should have been drawn under analysis at as early a time in the development of mechanics as that science became capable of dealing with continuous flexible bodies.

The finite discrete system of particles that most naturally approximates the continuous material string is suggested by the string of beads. More precisely, it is to be conceived of as comprised of any number n of equal concentrated mass particles, that are mounted respectively at equally spaced points along a string which is itself weightless, though strong, perfectly flexible, and elastic. Such "loaded strings" were used as the conceptual bases of a variety of significant investigations. Euler and Daniel Bernoulli based upon them, in 1732 or 1733, studies of the motions of heavy dangling chains, and Euler, among other things, regarding the particles as oscillating longitudinally, built upon them in 1746 a theory of sound. [2] Mechanically simpler and chronologically even earlier than these researches, were certain investigations of John Bernoulli upon the transverse vibrations of a string with its end points fixed. We find in a consideration of this problem a convenient point of departure for our present discussion.

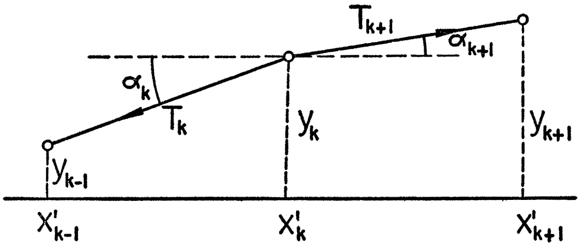
Consider, therefore, a loaded string such as has been described, with loading particles having a total mass M. Let this string be thought of as held taut under a tension T, the magnitude of this being so large that the ratio M/T is negligible. This last stipulation amounts, of course, to an idealization. Substantially its purpose is to discount the effects of gravity, and thus to concentrate the considerations upon those forces which spring from the tension alone. The divergence from physical actuality which this simplifying hypothesis sanctions is in many important instances of a very small amount. In the case of musical strings, for example, the ratio M/T which is to be ignored is quite commonly of a magnitude no greater than one one-thousandth.

The initial state of the string in question is to be one of displacement from its equilibrium position, the forces which hold it in this state being coplanar and directed perpendicularly to the line through the string's fixed end points. The particles of the string thus lie along some plane curve. We shall choose the plane of this curve as the (x, y) plane, and shall take the x-axis through the string's end points with the origin at one of them. Under the hypotheses made, and in this system of reference, the x-axis then marks the string's equilibrium position. With l designating the length of the string, with $x_0 = 0$ its initial end, and with $x_1, x_2, \cdots, x_n$ the equilibrium abscissas of the particles, the formulas for these are to be

$$x_k = k\frac{l}{n}, \qquad k = 0, 1, 2, \cdots, n.$$

It will be noted that in this assignment one of the particles is allocated to be mounted at the point $x_n = l$ which is the terminal end of the string. The motivation for adopting this arrangement, which may well seem a bit curious, is not by any means profound. It is merely one having some formal advantages, since, as can be shown, it leads to somewhat simpler formulas. The rôle of the nth particle under these circumstances is, of course, an entirely passive one since the particle is constrained from all motion.

In the accompanying figure three adjacent particles and the tensions operating between them are schematically indicated. The position shown is one of displacement from the equilibrium, and the particle which in the latter state is located at the point $(x_k, 0)$ is here shown to have the coördinates (x_k', y_k). Simple mechanical considerations based upon this figure readily lead to usable results, as we now propose to show.



The segment of the string which joins the $(k-1)$th and the kth particles is, as is clear from the figure, of the length $(x_k' - x_{k-1}')$ sec α_k. In the state of equilibrium this segment would be of the length l/n, and since, by the laws of elasticity, the respective lengths are to each other as the tensions, it must be concluded that

$$(x_k' - x_{k-1}') \sec \alpha_k : \frac{l}{n} = T_k : T,$$

or

$$(x_k' - x_{k-1}')Tn = lT_k \cos \alpha_k. \tag{3.1}$$

Now the forces by which the particles are initially held displaced are by hypothesis in the direction of the y-axis. The tensions in the string segments which balance them must, therefore, have x-components which annul each other in pairs, namely they must be such that

$$T_k \cos \alpha_k = T_{k+1} \cos \alpha_{k+1}, \qquad k = 1, 2, \cdots, (n-1). \tag{3.2}$$

Since the right-hand members of the equations (3.1) are thus all equal, this equality must extend to the left-hand members as well. The differences $(x_k' - x_{k-1}')$ therefore have a common value, and since their sum is the length of the string this value is evidently l/n. It follows that for each k the relation $x_k' = x_k$ maintains, namely that each particle has when displaced the same abscissa as in equilibrium. The measures of the displacements are thus simply the ordinates y_k, and the relations (3.1) reduce to the forms

$$T_k \cos \alpha_k = T, \qquad k = 1, 2, \cdots, (n-1). \tag{3.3}$$

From the initial position which has been described the string is now to be thought of as released, while in the state of rest, at an instant which is to be taken as the origin of time, $t=0$. The motion into which the kth particle springs is, of course, then governed by Newton's law

$$\frac{M}{n}\frac{d^2y_k}{dt^2} = F_k,$$

the force F_k acting upon it being shown by the figure to have the value

$$F_k = T_{k+1}\sin\alpha_{k+1} - T_k\sin\alpha_k.$$

By virtue of the relations (3.3) this formula is alternatively expressible as

$$F_k = T(\tan\alpha_{k+1} - \tan\alpha_k),$$

and in terms of the coördinates this is

$$F_k = T\left[\frac{(y_{k+1} - y_k) - (y_k - y_{k-1})}{(l/n)}\right].$$

Hence if the constant a^2 is defined by the relation

$$a^2 = \frac{lT}{M}, \tag{3.4}$$

the equations of motion are

$$\frac{d^2y_k}{dt^2} = \left(\frac{na}{l}\right)^2[y_{k+1} - 2y_k + y_{k-1}], \qquad k = 1, 2, \cdots, (n-1). \tag{3.5}$$

In many respects the simplest modes of vibration which the string is capable of are those in which the ordinates y_k maintain constant ratios to each other. In these so-called *normal* vibrations all particles of the string traverse their positions of equilibrium in synchronism, and their displacements are expressible as functions of the time by formulas of the type

$$y_k(t) = u_k\phi(t), \qquad k = 0, 1, 2, \cdots, n, \tag{3.6}$$

in which the coefficients u_k are constants, with $u_0 = 0$, $u_n = 0$, and $\phi(t)$ is common to them all. The fact that the motion originates from the state of rest is then expressed by the relation $\phi'(0) = 0$.

Now the substitution of the forms (3.6) into the equations (3.5) gives to these latter the aspect

$$u_k\phi''(t) = \left(\frac{na}{l}\right)^2[u_{k+1} - 2u_k + u_{k-1}]\phi(t).$$

From this it is clear that the second derivative $\phi''(t)$ stands in a constant ratio to the function $\phi(t)$ itself, namely that

$$\frac{d^2\phi(t)}{dt^2} = -c^2\phi(t), \qquad \phi'(0) = 0, \tag{3.7}$$

the constant c being one for which the relations

$$u_{k+1} - 2u_k + u_{k-1} = -\left(\frac{cl}{na}\right)^2 u_k, \qquad k = 1, 2, \cdots, (n-1),$$

together with $u_0 = 0$, $u_n = 0$ maintain. In terms of the coefficient q given by the formula

$$q = \left(\frac{cl}{na}\right)^2 - 2, \tag{3.8}$$

the values u_k must thus be solutions of the algebraic system of equations

$$\begin{aligned} u_0 &= 0, \\ u_{k+1} + qu_k + u_{k-1} &= 0, \qquad k = 1, 2, \cdots, (n-1), \\ u_n &= 0. \end{aligned} \tag{3.9}$$

This system is neatly solvable (*c.f.* appendix I) having a non-trivial solution when and only when the coefficient q has one of the set of characteristic values $q_1, q_2, \cdots, q_{n-1}$, given by the formulas (I.5). The values of c which respectively correspond to these under the relation (3.8) are those of the set

$$c_\nu = \frac{2na}{l} \sin \frac{\nu\pi}{2n}, \qquad \nu = 1, 2, \cdots, (n-1), \tag{3.10}$$

and the solution $u_{\nu,k}$ of the system (3.9) which exists for the value c_ν is obtainable from the formulas (I.6). It is

$$u_{\nu,k} = A_\nu \sin \frac{k\nu\pi}{n}, \qquad k = 0, 1, 2, \cdots, n,$$

with A_ν designating any constant. Since when $c = c_\nu$ the equations (3.7) are solved by the function

$$\phi(t) = \cos (c_\nu t),$$

or by a constant multiple of this, it may be drawn from the relations (3.6) that the ordinates in any normal vibration of the loaded string must accord with the formulas

$$y_k(t) = A_\nu \sin \frac{k\nu\pi}{n} \cos \left(\frac{2ant}{l} \sin \frac{\nu\pi}{2n}\right), \qquad k = 0, 1, 2, \cdots, n. \tag{3.11}$$

A loaded string carrying n particles is thus seen to be capable of sustaining $(n-1)$ distinct motions of the normal type, these being given by the formulas (3.11) in correspondence with the indices $\nu = 1, 2, \cdots, (n-1)$.

Under assumptions that were somewhat more restrictive than those which we have here imposed, these normal vibrations were considered by John Bernoulli as early as the year 1728 in the cases of loaded strings in which particles up to eight in number were involved. [3]

CHAPTER 4

The equations of motion for the continuous string. When the differential equations (3.5) for the motions of the loaded string of n particles have once been deduced, two alternative modes of procedure for utilizing them toward the ultimate purpose of an analysis of the vibrations of a continuous string suggest themselves. On the one hand the equations may be integrated, as has already been done in the preceding chapter, and the resulting finite equations (3.11) may then be subjected to the limiting process in which the number n is indefinitely increased. On the other hand this limiting process may be applied directly to the system of equations (3.5) itself, and the integration of the result may then subsequently be undertaken. Both of these procedures were carried out in the first half of the eighteenth century. As we shall see, their results are of quite dissimilar aspects. Indeed they seemed to the men of the time to be no less than contradictory, to the extent that the proponents of either method saw no alternative but to reject the other. That no real dilemma was actually involved therein at all, came to its realization only half a century or more later. The clarifications of ideas by which the way out of the quandary was ultimately found are of especial interest to us here. We propose, therefore, to pursue the analysis of the two mentioned procedures to such points, at least, as afford some surveys of their conclusions.

Returning then, to begin with, to the equations (3.11), let any positive integer ν be chosen. Once chosen, ν is to be regarded as fixed. A loaded string with particles in number greater than ν, $(n>\nu)$, may then be thought of, and for the normal motions of such a string the equations (3.11) are derivable. Let the attention then be fixed upon any one of the particles of this string, and let its abscissa and ordinate be designated by x and $y(t, x)$. If this particle, in the enumeration that was adopted, is the kth one, the equalities

$$x = k\frac{l}{n}, \qquad y(t, x) = y_k(t), \tag{4.1}$$

evidently maintain. The respective kth equation of the set (3.11) may then be written in the manner

$$y(t, x) = A_\nu \sin\frac{\nu\pi x}{l}\cos\left[\theta_n\frac{\nu\pi a t}{l}\right], \tag{4.2}$$

with the significance of θ_n given by the formula

$$\theta_n = \frac{\sin\left(\frac{\nu\pi}{2n}\right)}{\left(\frac{\nu\pi}{2n}\right)}. \tag{4.3}$$

Suppose now that the parameters n and k are increased, and indefinitely so, in any way such that the ratio k/n remains fixed. The point x then clearly remains invariant, and inasmuch as the formula (4.3) familiarly shows that the value θ_n approaches the limit 1 it follows that the relation (4.2) passes, as $n \to \infty$, into the limiting form

$$y(t, x) = A_\nu \sin \frac{\nu \pi x}{l} \cos \frac{\nu \pi a t}{l} \cdot \tag{4.4}$$

This result is now evidently to be accepted as a formula applicable to the continuous string and representing a normal vibration of it. Inasmuch as the integer ν was initially open to an arbitrary choice, the inference that infinitely many such normal motions are possible and that they are given by the formulas (4.4) in conjunction with the indices $\nu = 1, 2, 3, \cdots$ is inevitable.

The simplest of the normal vibrations, namely that described by the formula (4.4) with $\nu = 1$ was deduced by Brook Taylor at as early a date as 1713. [4] In this motion the string vibrates without nodes and emits its fundamental tone. The existence of other normal motions, namely those associated by the formula (4.4) with other values of ν and in which the string emits its various over-tones, were known later to Daniel Bernoulli. We shall have occasion to return to this matter again.

The alternative procedure, to which we now turn, is associated most prominently with the names of d'Alembert and Euler. With the notational changes (4.1) and with the definition of Δx by the formula $\Delta x = l/n$, the kth one of the equations (3.5) may evidently be written in the form

$$\frac{\partial^2 y(t, x)}{\partial t^2} = a^2 \left[\frac{y(t, x + \Delta x) - 2y(t, x) + y(t, x - \Delta x)}{(\Delta x)^2} \right]. \tag{4.5}$$

Now whenever the function $y(t, x)$ is one which is twice differentiable as to x, its second partial derivative with respect to x is obtainable as the limit of the difference quotient within brackets on the right of the equality (4.5) as $\Delta x \to 0$, namely as $n \to \infty$. Basing himself upon this observation d'Alembert deduced in 1747 the partial differential equation

$$\frac{\partial^2 y(t, x)}{\partial t^2} = a^2 \frac{\partial^2 y(t, x)}{\partial x^2}, \tag{4.6}$$

for the motion of the continuous string. This result is of course still a standard. A solution of it, if it is to represent the ordinates of a string that is fastened at its end points upon the x-axis and that springs into motion at the time $t = 0$ from the state of rest and from the position of a curve $y = f(x)$, must, of course, also fulfill the conditions

$$
\begin{aligned}
y(t, 0) &= 0, \\
y(t, l) &= 0, \\
\left.\frac{\partial y(t, x)}{\partial t}\right]_{t=0} &= 0, \\
y(0, x) &= f(x).
\end{aligned}
\tag{4.7}
$$

The problem as thus formulated was solved by d'Alembert through the following ingenious use of familiar formulas from the calculus. [5]

In terms of the abbreviations

$$
\frac{\partial y}{\partial t} = p, \qquad \frac{\partial y}{\partial x} = q,
\tag{4.8}
$$

the differential equation (4.6) is expressible in the form

$$\frac{\partial q}{\partial x} = \frac{1}{a^2} \frac{\partial p}{\partial t},$$

whereas it is familiar that generally

$$\frac{\partial q}{\partial t} = \frac{\partial p}{\partial x}.$$

By the use of these relations the standard formula

$$dq = \frac{\partial q}{\partial t}\,dt + \frac{\partial q}{\partial x}\,dx,$$

may, however, be written thus

$$dq = \frac{\partial p}{\partial x}\,dt + \frac{1}{a^2} \frac{\partial p}{\partial t}\,dx.$$

From this together with the companion formula

$$dp = \frac{\partial p}{\partial t}\,dt + \frac{\partial p}{\partial x}\,dx,$$

it may be seen at once that

$$
\begin{aligned}
d\left(\frac{p}{a} + q\right) &= \left[\frac{1}{a} \frac{\partial p}{\partial x} + \frac{1}{a^2} \frac{\partial p}{\partial t}\right] d(at + x), \\
d\left(\frac{p}{a} - q\right) &= \left[\frac{-1}{a} \frac{\partial p}{\partial x} + \frac{1}{a^2} \frac{\partial p}{\partial t}\right] d(at - x).
\end{aligned}
\tag{4.9}
$$

Consider the first one of these equations. The quantity $(p/a+q)$ is a function of t and x. These variables are in turn determinable from the combinations

$(at+x)$ and $(at-x)$. It follows from this that it is permissible to regard the quantity as a function of the variables $(at+x)$ and $(at-x)$, and that accordingly its differential is given by the formula

$$d\left(\frac{p}{a}+q\right)=\frac{\partial\left(\dfrac{p}{a}+q\right)}{\partial(at+x)}\,d(at+x)+\frac{\partial\left(\dfrac{p}{a}+q\right)}{\partial(at-x)}\,d(at-x).$$

But by the evaluation (4.9) this differential includes no term in $d(at-x)$. The coefficient of this term must, therefore, be zero, namely

$$\frac{\partial\left(\dfrac{p}{a}+q\right)}{\partial(at-x)}=0.$$

This, however, is in effect the assertion that the quantity $(p/a+q)$ does not depend upon the variable $(at-x)$ but is a function of the remaining variable $(at+x)$ alone. A similar chain of reasoning shows that the quantity $(p/a-q)$ is a function of the variable $(at-x)$ alone, namely that with appropriate functions designated by ϕ and ψ,

$$\frac{p}{a}+q=\phi(at+x),$$

$$\frac{p}{a}-q=\psi(at-x).$$

If these relations are now multiplied respectively by the factors $(adt+dx)/2$ and $(adt-dx)/2$ and are then added, the result on the left of the equality is $pdt+q\,dx$, which is identified as dy. Thus

$$dy=\tfrac{1}{2}\phi(at+x)d(at+x)+\tfrac{1}{2}\psi(at-x)d(at-x).$$

In this formula each term is an exact differential. An integration is, therefore, possible and shows that

$$y(t,x)=\tfrac{1}{2}\Phi(at+x)+\tfrac{1}{2}\Psi(at-x), \tag{4.10}$$

the functions Φ and Ψ being indefinite integrals of ϕ and ψ respectively. With the attainment of the result (4.10) d'Alembert had deduced the fact that every solution of the partial differential equation (4.6) is of necessity expressible as the sum of a function of the variable $(at+x)$ and a function of the variable $(at-x)$. It is a simple matter to show conversely, by direct substitution, that if Φ and Ψ are any suitably differentiable functions, the formula (4.10) does in fact give a solution of the differential equation.

To apply to the particular case of the vibrating string the solution (4.10) must furthermore conform to the conditions (4.7). Of these the first one, which when applied to the relation (4.10) assumes the form

$$\tfrac{1}{2}\Phi(at) + \tfrac{1}{2}\Psi(at) = 0,$$

shows that the function Ψ must be identical with $-\Phi$. That being so the second condition takes the form

$$\tfrac{1}{2}\Phi(at + l) = \tfrac{1}{2}\Phi(at - l), \tag{4.11}$$

and since this is to be an identity in t it shows that the function Φ must be periodic with the span $2l$ as a period. The third condition (4.7) reduces to the relation

$$\Phi'(x) = \Phi'(-x).$$

Upon an integration this becomes

$$\Phi(x) = -\Phi(-x), \tag{4.12}$$

and thus characterizes Φ to be an odd function. The last condition, which must maintain over the string's length, reduces then to the relation

$$\Phi(x) = f(x), \qquad 0 \leqq x \leqq l. \tag{4.13}$$

In total it is to be concluded, therefore, that for the vibrating string

$$y(t, x) = \tfrac{1}{2}\Phi(at + x) - \tfrac{1}{2}\Phi(at - x), \tag{4.14}$$

every motion of the string being so representable with an appropriate function Φ. In the instance of any particular motion, in which the curve from which the string is released is $y = f(x)$, the function Φ that is concerned is determined over the interval $(0, l)$ by the relation (4.13) and is defined for all other arguments by its character of being odd and periodic.

CHAPTER 5

The d'Alembert-Euler-Bernoulli controversy. [6] The method of d'Alembert in his analysis of the problem of the vibrating string was also the method chosen by Euler. Superficially, therefore, the initial memoirs of these masters, written, as they were at short intervals of each other, differed mainly in their details. In their over-all aspect they resembled each other markedly, at least insofar as their formal features were concerned. Only below the surface did the lines of thought show themselves to be divergent, as sharply so, at points, as were the two men in the characters of their genius. Euler's temperament was an imaginative one. He looked for guidance in large measure to practical considerations and physical intuition, and combined with a phenomenal ingenuity an almost naive faith in the infallibility of mathematical formulas and the results of manipulations upon them. D'Alembert was a more critical mind, much less susceptible to conviction by formalisms. A personality of impeccable scientific integrity, he was never inclined to minimize short-comings that he recognized, be they in his own work or in that of others.

To Euler the solution of the problem of the string seemed definitive. Since any and every motion originating from the state of rest would necessarily stem from some initial shape of the string, and since he was willing to accept as the function $f(x)$ any distribution of values consistent with such a shape physically realizable, he maintained the solution of d'Alembert and himself to be the completely general one. The values $f(x)$ involved in the formula (4.13) he regarded as appropriately subject, if necessary, to graphical definition. To such interpretations d'Alembert took exception. He regarded the functional symbol as standing for an expression which could be constructed by the ordinary processes of algebra and the calculus from the independent variables. In having taken the ordinates of the string to be denotable in the form $y(t, x)$, he believed that the results could apply only to such motions as might be characterized by the fact that in them the string shapes at any two instants t_1 and t_2 are obtainable from one and the same formal expression $y(t, x)$ by giving to t the respective values. He saw no reason to suppose that all possible motions conform to this. Furthermore, inasmuch as the differential equation from which the solution emerges involves the derivative $\partial^2 y/\partial x^2$, he was unwilling to admit the applicability of the analysis to cases in which the function $f(x)$ is not twice differentiable. Finally, because of the relation (4.13) he insisted upon restricting the solution to instances in which the function $f(x)$ is periodic. That there might conceivably exist expressions $\Phi(x)$ and $f(x)$ yielding the same values over some specific interval but not persisting in this relationship for other values of the variable, was believed by neither d'Alembert nor Euler nor by any of their contemporaries.

A difference between d'Alembert and Euler lay in the fact that whereas the former was inclined to look upon the concepts of the function and the analytic expression as synonymous, the latter would not hold to this. Euler saw no reason, for instance, to rule out the possibility of releasing a string from the position of a curve made up of circular arcs of different radii, provided these arcs joined with each other continuously and with a continuously turning tangent line. It is evident that Euler had advanced measurably to the conception of an *arbitrary* curve. It is understandable, however, that d'Alembert should have declined to acknowledge the legitimacy of admitting such curves into consideration where the operations of the calculus were to be employed.

In 1755 a memoir of Daniel Bernoulli's upon the motions of the string turned the entire disagreement into new channels. Bernoulli, who had interested himself in acoustics, had recognized the relation between the several normal vibrations and the respective overtones which the string could be made to emit. It was a generally recognized fact at the time that a musical string ordinarily responds with a combination of its fundamental and overtones. Bernoulli had discovered that the motions involved in this do, in a very definite sense, retain their individuality—that in the entire motion the several normal vibrations are simply superposed upon each other. It was a relatively moderate step from this to the conception that all possible motions of a string are but linear combinations of the normal vibrations, variations in the relative intensities of the overtone components producing the observable differences in the timbre of the tone. In terms of symbols, and with the use of the formulas (4.4), this comes to its formulation in the assertion that every motion of the string is expressible in the form

$$y(t, x) = \sum_{\nu=1}^{\infty} A_\nu \sin \frac{\nu\pi x}{l} \cos \frac{\nu\pi a t}{l}, \tag{5.1}$$

with appropriate constant coefficients A_ν.

Neither Euler nor d'Alembert was inclined to accept this, and each made his rejoinder. Euler quickly recognized the fact that a motion representable in the form (5.1) would be one for which the initial ordinates have the values

$$\sum_{\nu=1}^{\infty} A_\nu \sin \frac{\nu\pi x}{l} \cdot \tag{5.2}$$

To assume that all motions are here involved would, he pointed out, come to the assertion that an arbitrary function $f(x)$ could be represented by a series of the type (5.2). Since among other things any expression (5.2) is odd and periodic, it seemed to Euler that he had reduced Bernoulli's claim to a manifest absurdity. That Bernoulli's result gave solutions—special ones—he did not deny. He had, in fact, made that discovery on his own account some years earlier.

D'Alembert, on his part, not only endorsed all of Euler's objections but went in fact well beyond them. He was unwilling to concede even that any and every odd and suitably periodic function could be represented by an expression (5.2), maintaining, in particular, that the function would need to be twice differentiable since that is so of all terms of the series. Bernoulli's analysis, and especially his passage from the finite case of the loaded string to the continuous one, had been at best sketchy and fragmentary. His opponents found much that could properly be rejected in that.

On the whole the objections left Bernoulli unshaken. In replying to Euler's claim of absurdity he referred to the fact that any finite sum of m terms from the expression (5.2) could, by an appropriate determination of the coefficients, be made to coincide in value with any given function $f(x)$ at any chosen set of points m in number. He saw no reason, therefore, for rejecting the possibility that the series (5.2), involving infinitely many coefficients as it does, might not coincide with an arbitrary function at an infinity of points. This viewpoint was indeed a worthy one. A development of it will concern us in the following chapter.

The three cornered polemic spreads itself through the mathematical literature over a period of more than a decade. Since no one of the contenders succeeded in convincing another, the upshot of the matter at the time was negligible. Each of the disputants was in part right and in part wrong. Time has given the lion's share of its endorsements to Bernoulli.

CHAPTER 6

Lagrange's solution of Bernoulli's problem in curve fitting. [7] The fact invoked by Bernoulli, that an arbitrarily given curve is representable at any finite set of abscissas by a suitable segment of a series (5.2), is one of considerable importance both from the theoretical and practical standpoints. It devolves, of course, upon the possibility of determining the coefficients c_k in a formula

$$y = \sum_{k=1}^{n-1} c_k \sin \frac{k\pi x}{l},$$

so that the curve here represented may pass through a prescribed set of points (x_ν, F_ν), $\nu = 1, 2, \cdots, (n-1)$, the abscissas of which lie upon the interval $(0, l)$. Alternatively stated it comes to the fact that with an arbitrary assignment of constants F_ν the system of equations

$$\sum_{k=1}^{n-1} c_k \sin \frac{k\pi x_\nu}{l} = F_\nu, \qquad \nu = 1, 2, \cdots, (n-1), \tag{6.1}$$

is solvable for the values c_k.

The solution of any linear algebraic system of equations, and hence in particular of this system, is, of course, possible by elementary procedures. Such a frontal attack upon it by the familiar method of determinants leads, however, through much tedious and protracted computation whenever the number of equations is large. The solution, moreover, is not likely to emerge from such manipulations in any neat or elegant form. At the very beginning of his career, while he was still in his early twenties, Lagrange concerned himself with this problem and gave solutions of it for both the cases in which the abscissas are equally and unequally spaced. It is the former of these which is of peculiar pertinence to our discussion, and although it appears in Lagrange's work incidentally to the wider investigation with which we shall be concerned in the next chapter, a self-contained exposition of it is possible and is to be given here. It is a prime merit of this solution that it shows clearly how it depends upon the number n of points involved, and that it is therefore excellently adapted to an investigation in which this number is ultimately to be varied and to be allowed to become infinite.

Let the abscissas x_ν be identified again thus

$$x_\nu = \frac{\nu l}{n}, \qquad \nu = 0, 1, 2, \cdots, n, \tag{6.2}$$

and let any one of the integers $1, 2, \cdots, (n-1)$ be chosen and designated j. If the equations (6.1) are multiplied by the respective constant of an undetermined set $D_{j,\nu}$ and are then added, the result is the relation

$$\sum_{\nu=1}^{n-1}\sum_{k=1}^{n-1} D_{j,\nu}c_k \sin\frac{k\pi x_\nu}{l} = \sum_{\nu=1}^{n-1} D_{j,\nu}F_\nu.$$

Since $kx_\nu = \nu x_k$ this can be given the alternative form

$$\sum_{k=1}^{n-1} \Phi_j(x_k)c_k = \sum_{\nu=1}^{n-1} D_{j,\nu}F_\nu, \tag{6.3}$$

the function Φ_j herein being defined by the formula

$$\Phi_j(x) \equiv \sum_{\nu=1}^{n-1} D_{j,\nu} \sin\frac{\nu\pi x}{l}\cdot \tag{6.4}$$

Now each function sin $(\nu\pi x/l)$ in this relation is expressible as the product of sin $(\pi x/l)$ by a polynomial of the degree $(\nu-1)$ in cos $(\pi x/l)$ (*cf.* appendix II). The complete function Φ_j may therefore be similarly expressed, namely thus

$$\Phi_j(x) = \sin\frac{\pi x}{l}\, P_{n-2}\left(\cos\frac{\pi x}{l}\right), \tag{6.5}$$

with P_{n-2} designating a polynomial of the degree $(n-2)$. The coefficients of this polynomial depend, of course, upon the multipliers $D_{j,\nu}$, and these have not thus far been specified. It is proposed now to specify them so that the function $\Phi_j(x)$ may be zero at each of the points x_k with the specific exception of x_j, namely so that

$$\Phi_j(x_k) = 0, \qquad k \neq j. \tag{6.6}$$

Assuming this to be possible, it is clear from the equation (6.5) that each one of the $(n-2)$ values cos $(\pi x_\nu/l)$, $\nu\neq j$ must then be a root of P_{n-2}, and that each corresponding difference (cos $\pi x/l-$cos $\pi x_\nu/l$) must therefore be a factor. The factors are thus all accounted for, and with an appropriate constant α the formula (6.5) may accordingly be written

$$\Phi_j(x) = \alpha \sin\frac{\pi x}{l} \prod_{\nu=1,\nu\neq j}^{n-1}\left(\cos\frac{\pi x}{l} - \cos\frac{\pi x_\nu}{l}\right). \tag{6.7}$$

As is shown by the formula (II.1), however, a relation

$$\sin\frac{n\pi x}{l} = \sin\frac{\pi x}{l}\, p_{n-1}\left(\cos\frac{\pi x}{l}\right)$$

also maintains, and since the left-hand member of this is zero at each point x_ν without exception, the function p_{n-1}(cos $\pi x/l$) must admit as a factor each of the differences $(\cos \pi x/l - \cos \pi x_\nu/l)$. The factors being thus again all accounted for, it follows that

$$\sin \frac{n\pi x}{l} = \beta \sin \frac{\pi x}{l} \prod_{\nu=1}^{n-1} \left(\cos \frac{\pi x}{l} - \cos \frac{\pi x_\nu}{l} \right),$$

with β standing for some constant that is not zero. From this together with the evaluation (6.7) it appears that

$$\text{(6.8)} \qquad \left(\cos \frac{\pi x}{l} - \cos \frac{\pi x_j}{l} \right) \Phi_j(x) = \frac{\alpha}{\beta} \sin \frac{n\pi x}{l},$$

namely, because of the formula (6.4), that

$$\sum_{\nu=1}^{n-1} D_{j,\nu} \sin \frac{\nu\pi x}{l} \left(\cos \frac{\pi x}{l} - \cos \frac{\pi x_j}{l} \right) - \frac{\alpha}{\beta} \sin \frac{n\pi x}{l} = 0.$$

This equation may be reduced by the use of the familiar relation

$$\sin \frac{\nu\pi x}{l} \cos \frac{\pi x}{l} = \frac{1}{2} \sin \frac{(\nu+1)\pi x}{l} + \frac{1}{2} \sin \frac{(\nu-1)\pi x}{l},$$

and by a rearrangement of its terms, to appear in the form

$$\text{(6.9)} \qquad \sum_{k=1}^{n-1} [D_{j,k+1} + q_j D_{j,k} + D_{j,k-1}] \sin \frac{k\pi x}{l} + \left[D_{j,n-1} - \frac{2\alpha}{\beta} \right] \sin \frac{n\pi x}{l} = 0,$$

the coefficient q_j being specifically

$$\text{(6.10)} \qquad q_j = -2 \cos \frac{\pi x_j}{l},$$

and $D_{j,0}$ and $D_{j,n}$ being zero.

The equation (6.9) is identically fulfilled if the multipliers $D_{j,k}$ satisfy the system

$$D_{j,0} = 0,$$

$$D_{j,k+1} + q_j D_{j,k} + D_{j,k-1} = 0, \qquad k = 1, 2, \cdots, (n-1),$$

$$D_{j,n} = 0,$$

and furthermore

$$D_{j,n-1} = \frac{2\alpha}{\beta}.$$

This system is precisely that which is discussed in the appendix I, the coefficient (6.10) being that one of the values (I. 5) for which the system admits the solu-

(I. 6) with $\nu=j$. The free coefficient A_j in this may be determined, moreover, to yield the value prescribed above for $D_{j,n-1}$, and is thus found to have the value $2\alpha/\beta \sin (n-1)j\pi/l$, or because of the relations (6.2)

$$A_j = (-1)^{j+1} \frac{2\alpha}{\beta \sin \frac{\pi x_j}{l}}.$$

The evaluation of the multipliers which thus results is

$$D_{j,k} = (-1)^{j+1} \frac{2\alpha \sin \frac{k\pi x_j}{l}}{\beta \sin \frac{\pi x_j}{l}}, \tag{6.11}$$

and with these the equation (6.3) reduces by virtue of the values (6.6) to the form

$$\Phi_j(x_j)c_j = \sum_{\nu=1}^{n-1} D_{j,\nu}F_\nu. \tag{6.12}$$

It only remains, therefore, to determine the value of $\Phi_j(x_j)$, and this may be done as follows. The formula for $\Phi_j(x)$, as it is given by the equation (6.8), is indeterminate at $x=x_j$. By an application of l'Hospital's rule, however, its limiting value is found to be

$$\frac{n\alpha \cos \frac{n\pi x_j}{l}}{-\beta \sin \frac{\pi x_j}{l}},$$

namely because of the definitions (6.2),

$$\Phi_j(x_j) = (-1)^{j+1} \frac{n\alpha}{\beta \sin \frac{\pi x_j}{l}}.$$

This result, together with the evaluations (6.11), causes a final reduction of the relation (6.12) to the form

$$c_j = \frac{2}{n} \sum_{\nu=1}^{n-1} F_\nu \sin \frac{\nu\pi x_j}{l}, \tag{6.13}$$

and therewith the coefficients in the equations (6.1) have been determined.

CHAPTER 7

Lagrange and the vibrating string. [8] In the controversy over the problem of the vibrating string Lagrange was inclined on the whole to enlist himself upon the side of Euler. To support himself in this position he undertook to re-examine afresh the behavior of the weightless loaded string with an unspecified number of particles, his explicit purpose being to elicit from this a proof that in the case of the continuous string no restrictions upon the shape of the curve marking the initial position are requisite. His method in this has become a standard one. As do the deductions of the preceding chapter, it hinges primarily upon an introduction of undetermined multipliers. By this means he carried through, as we shall see, a general integration of the differential equations for the string's motion, and thus displayed in terms of explicit formulas the dependence of the string's position at any instant upon its initial shape.

The differential equations for the particles of the loaded string of the length l under the tension T, with n particles of total mass M located respectively at the points

$$x_k = \frac{kl}{n}, \qquad (k = 1, 2, \cdots, (n-1) \tag{7.1}$$

were deduced in chapter 3 and are given under (3.5). If the initial ordinates of the particles are denoted by f_k, and if the particles spring at $t=0$ from the state of rest in these positions, the boundary relations to which the differential equations are to be subjected are

$$\begin{aligned} y_k(0) &= f_k, \\ \left.\frac{dy_k(t)}{dt}\right]_{t=0} &= 0, \qquad k = 1, 2, \cdots, (n-1). \end{aligned} \tag{7.2}$$

Let the equations (3.5) be multiplied respectively by unspecified constants M_k. The addition of them then results in the single equation

$$\sum_{k=1}^{n-1} M_k \frac{d^2 y_k}{dt^2} = \left(\frac{na}{l}\right)^2 \sum_{k=1}^{n-1} M_k [y_{k+1} - 2y_k + y_{k-1}],$$

and this, under a re-grouping of its terms, together with the evaluations $M_0 = 0$, $M_n = 0$, takes on alternatively the form

$$\frac{d^2}{dt^2} \sum_{k=1}^{n-1} M_k y_k = \left(\frac{na}{l}\right)^2 \sum_{k=1}^{n-1} [M_{k+1} - 2M_k + M_{k-1}] y_k. \tag{7.3}$$

Consider now the possibility of so choosing the multipliers M_k as to make the corresponding terms of the two sums in the equation maintain a fixed ratio to each other. With a constant of proportionality γ, the condition upon the multipliers is then this, that they may comprise a solution of the linear algebraic system

$$(7.4)\qquad \begin{aligned} &M_0 = 0,\\ &M_{k+1} - 2M_k + M_{k-1} = \gamma M_k, \quad k = 1, 2, \cdots, (n-1),\\ &M_n = 0. \end{aligned}$$

This, however, is precisely the system (I. 1) of the appendix I, with $(-\gamma-2)$ in the rôle of the coefficient q. The values of γ for which the system is non-trivially solvable are thus found from the relations (I. 5) to be $(n-1)$ in number, namely $\gamma=\gamma_\nu$, $\nu=1, 2, \cdots, (n-1)$, with

$$(7.5)\qquad \gamma_\nu = -4\sin^2\frac{\nu\pi}{2n}\cdot$$

Upon designating by the symbols $M_{\nu,k}$ those multipliers M_k which satisfy the system when γ has the value (7.5), we find from the formulas (I. 6) the respective evaluations

$$(7.6)\qquad M_{\nu,k} = A_\nu \sin\frac{k\nu\pi}{n}, \qquad k = 0, 1, 2, \cdots, n,$$

the coefficient A_ν being arbitrary.

Let $\sigma_\nu(t)$ be used now as an abbreviation in the sense

$$(7.7)\qquad \sigma_\nu(t) \equiv \sum_{k=1}^{n-1} M_{\nu,k}y_k(t).$$

The differential equation (7.3) with its boundary relations (7.2) may then be written in the form

$$\frac{d^2\sigma_\nu}{dt^2} = \gamma_\nu\left(\frac{na}{l}\right)^2\sigma_\nu,$$

$$\sigma_\nu(0) = \sum_{k=1}^{n-1} M_{\nu,k}f_k,$$

$$\left.\frac{d\sigma_\nu}{dt}\right]_{t=0} = 0.$$

This is a differential system which is easily solvable by elementary means. Its solution is

$$\sigma_\nu(t) = \left(\sum_{k=1}^{n-1} f_k M_{\nu,k}\right)\cos\left(\sqrt{-\gamma_\nu}\,\frac{nat}{l}\right),$$

namely, in terms of the evaluations (7.5), (7.6), and (7.7),

$$(7.8) \qquad \sum_{k=1}^{n-1} y_k(t) \sin \frac{k\pi x_\nu}{l} = \sum_{k=1}^{n-1} f_k \sin \frac{\nu\pi x_k}{l} \cos\left(\frac{2nat}{l} \sin \frac{\nu\pi}{2n}\right).$$

Inasmuch as the index ν is free to take the values $1, 2, \cdots, (n-1)$, this is a system of $(n-1)$ equations.

In structure the system (7.8) is evidently of the form (6.1) with the values $y_k(t)$ in the place of unknowns c_k. Its solution is therefore given by the formulas (6.13) to be

$$(7.9) \qquad y_j(t) = \frac{2}{n} \sum_{\nu=1}^{n-1} \sum_{k=1}^{n-1} f_k \sin \frac{\nu\pi x_k}{l} \sin \frac{\nu\pi x_j}{l} \cos\left(\frac{2nat}{l} \sin \frac{\nu\pi}{2n}\right),$$
$$j = 1, 2, \cdots, (n-1).$$

With this result Lagrange's integration of the equations of motion is complete.

It is suggestive for our purposes to consider the formalisms of a passage from the discretely loaded string to the continuous one upon the basis of Lagrange's formulas. If in the equations (7.9) the notational changes indicated by the substitutions of x and $y(t, x)$ for x_j and $y_j(t)$, of s_k and $f(s_k)$ for x_k and f_k, and of Δs for l/n, are made, the formula assumes the aspect

$$y(t, x) = \frac{2}{l} \sum_{\nu=1}^{n-1} \left(\sum_{k=1}^{n-1} f(s_k) \sin \frac{\nu\pi s_k}{l} \Delta s \right) \sin \frac{\nu\pi x}{l} \cos \left[\frac{\nu\pi at}{l} \cdot \frac{\sin\left(\frac{\nu\pi}{2n}\right)}{\left(\frac{\nu\pi}{2n}\right)} \right].$$

As n is indefinitely increased the relation

$$\lim \frac{\sin\left(\frac{\nu\pi}{2n}\right)}{\left(\frac{\nu\pi}{2n}\right)} = 1,$$

maintains for each value of ν and from the very definition of the definite integral

$$\lim_{n\to\infty} \sum_{k=1}^{n-1} f(s_k) \sin \frac{\nu\pi s_k}{l} \Delta s = \int_0^l f(s) \sin \frac{\nu\pi s}{l}\, ds.$$

These relations suggest as the limiting form of the solution (7.9) the formula

$$(7.10) \qquad y(t, x) = \sum_{\nu=1}^{\infty} \left(\frac{2}{l} \int_0^l f(s) \sin \frac{\nu\pi s}{l}\, ds \right) \sin \frac{\nu\pi x}{l} \cos \frac{\nu\pi at}{l} \cdot$$

It will be noted at once that this is precisely of the type of the solution (5.1) for which Bernoulli had insisted upon holding out. In it the initial position of the string is along the curve whose ordinates are $f(x)$. At the instant $t=0$ the formula (7.10) thus reduces to the form

$$f(x) = \sum_{\nu=1}^{\infty}\left(\frac{2}{l}\int_0^l f(s)\sin\frac{\nu\pi s}{l}\,ds\right)\sin\frac{\nu\pi x}{l}\,. \tag{7.11}$$

The representability of an "arbitrary" function by a series of sines in precisely the manner (1.3), (1.4), would hereby seem to be unmistakably presaged.

Although Lagrange carried out a limiting analysis upon his formula (7.9), it varied in some respects from that outlined above. His comparative result was the relation

$$y(t,x) = \frac{2}{l}\int_0^l\left(\sum_{\nu=1}^{\infty}\sin\frac{\nu\pi s}{l}\sin\frac{\nu\pi x}{l}\cos\frac{\nu\pi a t}{l}\right)f(s)\,ds, \tag{7.12}$$

which differs from (7.10) in having the order of the integration and summation reversed. This form has the disadvantage of involving a series which is obviously divergent, a matter which was readily seized upon by opposing critics. Beyond that, both limiting considerations are open to criticism upon a number of accounts, for they fail to distinguish between the results of analytical operations upon an infinite series as a whole and upon the terms of the series individually. These distinctions, so essential to rigor, were but imperfectly understood at the time.

While in the manner shown the formula (7.9) could easily have led to the conclusion (7.11), it remains a fact that it did not do so. In deducing the form (7.12) Lagrange was bent upon a different purpose, wholly remote from that of proving any such a theory as would be implied by the relation (7.11). Indeed, when such a theory was announced by Fourier more than a half century later, the then aged Lagrange is said to have remained incredulous of it.

CHAPTER 8

Euler's determination of the coefficients. In the latter half of the eighteenth century the properties of trigonometrical series were very much to the fore of mathematical interest, and numerous memoirs were written during that time upon one phase or another of the subject of the representability of functions by means of such series. In the main, however, these papers advanced the general theory but little. They may well be left aside in the present discussion as of only subordinate interest. A conspicuous exception to this, however, is a work of Euler's which he appears to have written in the year 1777, although its publication was deferred until 1793, some years after his death. Concerned with functions known upon some grounds or other to be representable in terms of a cosine series of the type (1.5), Euler deduced in this work the formula (1.6) for the coefficients [9].

If $f(x)$ is any function which in terms of the variable ξ, under the relation $\xi=\cos \pi x/l$, is expansible for $-1 \leqq \xi \leqq 1$ in a convergent power series $\sum_{j=0}^{\infty} c_j \xi^j$, then $f(x)$ clearly admits of representation by a cosine power series of the form

$$(8.1) \qquad f(x) = \sum_{j=0}^{\infty} c_j \cos^j \frac{\pi x}{l},$$

over the interval $(0, l)$. For each term in this, however, there maintains a respective trigonometric identity (*cf.* appendix III), namely

$$\cos^j \frac{\pi x}{l} = \frac{1}{2^{j-1}} \sum_{\mu=0}^{[j/2]} \binom{j}{\mu} \cos \frac{(j-2\mu)\pi x}{l},$$

with $[j/2]$ designating the greatest integer not exceeding $j/2$. The substitution of these evaluations into the equation (8.1) and the subsequent collection of terms of like character, give the equation formally the aspect

$$(8.2) \qquad f(x) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} a_\nu \cos \frac{\nu \pi x}{l}.$$

This is the type (1.5). It will be clear that a rather extensive class of functions $f(x)$ fulfills the assumptions that are basic to this reasoning.

The argument given, although it is adequate to permit an inference of the *form* of the representation (8.2), is readily seen to be quite far from being practical. It yields neither an easily applicable nor a generally lucid method by which the coefficients a_ν therein may be quantitatively evaluated. Since the series converges, it may, of course, be inferred that

$$(8.3) \qquad \lim_{\nu \to \infty} a_\nu = 0.$$

From this slender source Euler succeeded in deducing an evaluation of the contants a_ν which is directly referable to the function $f(x)$ in question.

Let the symbol $n_{\sigma,\tau}$ for any integer n and any indices σ, τ, be defined to have the values

$$(8.4) \qquad n_{\sigma,\tau} = \begin{cases} n, & \text{if} \quad \sigma \equiv \tau \pmod{2n} \\ 0, & \text{if} \quad \sigma \not\equiv \tau \pmod{2n}. \end{cases}$$

There is, then, a trigonometric evaluation (*cf.* appendix IV) to the effect that

$$(8.5) \qquad \sum_{\mu=1}^{n-1} \cos \frac{\mu \sigma \pi}{n} = \frac{-1}{2} [1 + \cos \sigma \pi] + n_{\sigma,0}.$$

From this a certain related formula can be easily deduced. If the equation (8.5) is written successively with σ replaced by $(\nu+k)$ and by $(\nu-k)$, an addition of the results yields directly the equality

$$\sum_{\mu=1}^{n-1}\left[\cos\frac{\mu(\nu+k)\pi}{n}+\cos\frac{\mu(\nu-k)\pi}{n}\right]$$
$$= -\left[1+\tfrac{1}{2}\cos(\nu+k)\pi+\tfrac{1}{2}\cos(\nu-k)\pi\right]+n_{\nu+k,0}+n_{\nu-k,0},$$

and this is contractible in an obvious fashion into the formula

$$\sum_{\mu=1}^{n-1} 2\cos\frac{\mu k\pi}{n}\cos\frac{\mu\nu\pi}{n} = -\left[1+\cos k\pi\cos\nu\pi\right]+n_{\nu,k}+n_{\nu,-k}. \tag{8.6}$$

Consider now the representation (8.2). If this is multiplied by the factor $2\cos(\mu k\pi/n)$ and is then evaluated at $x=\mu l/n$, it yields the equalities

$$2f\left(\frac{\mu l}{n}\right)\cos\frac{\mu k\pi}{n} = a_0\cos\frac{\mu k\pi}{n}+\sum_{\nu=1}^{\infty} a_\nu 2\cos\frac{\mu\nu\pi}{n}\cos\frac{\mu k\pi}{n}\cdot$$

When these are summed with respect to μ the constants a_0 and a_ν appear with coefficients that are given by the formulas (8.5) and (8.6) respectively. It is thus found that

$$(8.7)\quad \begin{aligned} 2\sum_{\mu=1}^{n-1} f\left(\frac{\mu l}{n}\right)\cos\frac{\mu k\pi}{n} &= \frac{-a_0}{2}\left[1+\cos k\pi\right]-\sum_{\nu=1}^{\infty}a_\nu\left[1+\cos k\pi\cos\nu\pi\right] \\ &\quad + a_0 n_{k,0}+\sum_{\nu=1}^{\infty}a_\nu\left[n_{\nu,k}+n_{\nu,-k}\right]. \end{aligned}$$

This relation can be materially reduced. In the first place it will be found on the basis of the definitions (8,4) that for $k=0$, or for $k>0$ and $n>k$,

$$a_0 n_{k,0}+\sum_{\nu=1}^{\infty}a_\nu\left[n_{\nu,k}+n_{\nu,-k}\right] = n\left[a_k+\sum_{\lambda=1}^{\infty}\left(a_{2n\lambda-k}+a_{2n\lambda+k}\right)\right]\cdot$$

In the second place the formula (8.2) itself yields the equation

$$f(0)+f(l)\cos k\pi = \frac{a_0}{2}\left[1+\cos k\pi\right]+\sum_{\nu=1}^{\infty}a_\nu\left[1+\cos k\pi\cos\nu\pi\right].$$

With these evaluations, however, the formula (8.7) is simplified into the form

$$2\sum_{\mu=1}^{n-1}f\left(\frac{\mu l}{n}\right)\cos\frac{k\mu\pi}{n} = na_k-\left[f(0)+f(l)\cos k\pi\right]+n\sum_{\lambda=1}^{\infty}\left(a_{2n\lambda-k}+a_{2n\lambda+k}\right). \tag{8.8}$$

Let the notational substitutions $s_\mu=\mu l/n$, $\Delta s=l/n$, now be introduced. These, together with a division by n, give to the equation (8.8) the form

$$\frac{2}{l}\sum_{\mu=1}^{n-1}f(s_\mu)\cos\frac{k\pi s_\mu}{l}\Delta s = a_k-\frac{1}{n}\left[f(0)+f(l)\cos k\pi\right]+\sum_{\lambda=1}^{\infty}\left(a_{2n\lambda-k}+a_{2n\lambda+k}\right).$$

In this n is now to be permitted to become infinite. Since each term $a_{2n\lambda\pm k}$ in the final sum approaches zero by virtue of the relation (8.3), and the term in $1/n$ does likewise, the right-hand member of the equation has as its limit a_k. The limit of the left-hand member being a definite integral, the conclusion is that

$$\frac{2}{l}\int_0^l f(s)\cos\frac{k\pi s}{l}\,ds = a_k. \tag{8.9}$$

Herewith the problem was solved.

Once in possession of the formula (8.9), Euler recognized the more direct manner in which he might have found it, and by means of which a verification of it might be made. This is, namely, the procedure now generally familiar, of multiplying the representation (8.2) by the factor cos $(k\pi x/l)$, integrating it then term by term and applying the elementary evaluations

$$\int_0^l \cos\frac{\nu\pi x}{l}\cos\frac{k\pi x}{l}\,dx = \begin{cases} 0, & \text{if } \nu \neq k, \\ l/2, & \text{if } \nu = k \neq 0, \\ l, & \text{if } \nu = k = 0. \end{cases} \tag{8.10}$$

To this day the constants a_k as given by the formulas (8.9) are still widely known as the "Euler coefficients" of the function $f(x)$. Such an attribution seems, however, to be somewhat over-generous if representations (8.2) of *arbitrary* functions are in question. Euler was consciously concerned only with such functions as were known upon other grounds to be representable in a cosine series. The crucial observation that the formulas (8.9) are significant for functions of a much wider class than those which, for instance, are representable in the manner (8.1) apparently escaped him. There is no evidence, either in this connection or in any other, that he ever receded from his opposition to Bernoulli's claim that arbitrary functions submit to trigonometric representation

CHAPTER 9

Fourier and the theory of heat. In the interior of a material body heat is in general distributed in a manner that is both non-uniform and fluctuating—that is to say with temperatures that vary from point to point and from time to time. The distribution of temperatures throughout a body is, therefore, naturally determined by a function of the coördinates of position and time. What the precise form of this function is, in any particular case, depends in part upon the thermal properties of the material of which the body is constituted—its density, specific heat, and conductivity—but also in large part upon the instantaneous state in which the body finds itself at some specific time, and upon the conditions which thereafter maintain upon its surface.

In the early years of the nineteenth century Fourier devoted himself to an analysis of this temperature function τ, and deduced from physical fundamentals [10] the fact that it must satisfy a partial differential equation of the form

$$\nabla^2\tau = \kappa^2 \frac{\partial\tau}{\partial t}. \tag{9.1}$$

In this κ^2 is a positive constant whose value is determined by the thermal properties of the material, whilc $\nabla^2\tau$ is the so-called "Laplacian of τ." In terms of rectangular coördinates this differential expression is

$$\frac{\partial^2\tau}{\partial x^2} + \frac{\partial^2\tau}{\partial y^2} + \frac{\partial^2\tau}{\partial z^2},$$

if all three of the coordinates x, y, z, are significant, or, more simply

$$\frac{\partial^2\tau}{\partial x^2} + \frac{\partial^2\tau}{\partial y^2} \quad \text{or} \quad \frac{\partial^2\tau}{\partial x^2}$$

respectively, if the only space coördinates are x, y, or merely x alone. A temperature function must accordingly solve a partial differential equation such as (9.1). In any specific instance it must be *that* solution of this equation which takes on those values which apply at some specific instant t, and which furthermore fulfills upon the body's surface the thermal relations that maintain there.

A simple and familiar physical formulation illustrates this and will serve also to show the relevancy of this subject of the flow of heat to the basic matter before us, namely that of the representation of arbitrary functions by the means of trigonometric series. Consider a homogeneous material bar in the shape of a right cylinder of the length l and of any cross section. We may choose our coördinate system so that the direction of this bar is that of the x-axis with the end faces of the bar located at the points $x=0$ and $x=l$. Let it be supposed now that at some specific instant, which may be designated as $t=0$, the temperatures at all points within the bar having the abscissa x have the common value $f(x)$. From this instant onward each end face of the bar is to be held constantly at the temperature zero, while the lateral surface is insulated against the passage of heat. Except in the trivial case in which $f(x)$ is everywhere zero, heat will flow within the bar, and the lines of flow will be parallel to the x-axis. The problem is to determine the temperature at any point of the bar at any instant subsequent to the initial one, namely to determine the function $\tau(x, t)$ for $0<x<l$, and $t>0$. The relations from which this is to be done are in this case evidently the differential equation

$$\frac{\partial^2 \tau}{\partial x^2} = \kappa^2 \frac{\partial \tau}{\partial t}, \tag{9.2}$$

the boundary relations

$$\begin{aligned} \tau(0, t) &= 0, \\ \tau(l, t) &= 0, \qquad t > 0, \end{aligned} \tag{9.3}$$

and the initial condition

$$\tau(x, 0) = f(x), \quad 0 < x < l. \tag{9.4}$$

Fourier's method of attack upon a problem such as this is one that is still widely current in practice, namely the method of the "separation of variables." Let us, to begin with, seek a function $\tau_\nu(x, t)$ to fulfill the equations (9.2) and (9.3), while being of the form of a product of a function of the single variable x by a function of the single variable t, namely

$$\tau_\nu(x, t) = \phi_\nu(x)\psi_\nu(t). \tag{9.5}$$

Upon substituting this form into the equations in question, it is found that these are satisfied if the function $\phi_\nu(x)$ fulfills the system of relations

$$\begin{aligned} &\text{(i)} && \frac{d^2}{dx^2}\phi_\nu(x) + \lambda_\nu \kappa^2 \phi_\nu(x) = 0, \\ &\text{(ii)} && \phi_\nu(0) = 0, \\ &\text{(iii)} && \phi_\nu(l) = 0, \end{aligned} \tag{9.6}$$

with λ_ν designating a constant, provided that with this same constant the function $\psi_\nu(t)$ fulfills the equation

$$\frac{d}{dt}\psi_\nu(t) + \lambda_\nu \psi_\nu(t) = 0. \tag{9.7}$$

As a solution of the ordinary differential equation (9.6i) the function $\phi_\nu(x)$ must familiarly be of the form

$$\phi_\nu(x) = b_\nu \sin \sqrt{\lambda_\nu}\, \kappa x + \gamma_\nu),$$

in which b_ν and γ_ν may be any constants. It will fulfill the condition (9.6ii) if $\gamma_\nu = 0$, and then also the condition (9.6iii) if $\sqrt{\lambda_\nu}$ is any multiple of the constant $\pi/\kappa l$, namely, when

$$\lambda_\nu = \left(\frac{\nu\pi}{\kappa l}\right)^2, \tag{9.8}$$

with ν an integer. The associated solution of the equation (9.7) is then clearly

$$\psi_\nu(t) = e^{-(\nu^2\pi^2/\kappa^2 l^2)t}.$$

Through the formula (9.5) it has thus been found that the equations (9.2) and (9.3) are satisfied by the function

$$\tau_\nu(x, t) = b_\nu e^{-(\nu^2\pi^2/\kappa^2 l^2)t} \sin \frac{\nu\pi x}{l}, \tag{9.9}$$

in fact by each of the infinite set of functions obtained from this formula by setting $\nu = 1, 2, 3, \cdots$.

Now it is a characteristic property of linear equations or of any system of such, that the sum of any set of solutions is itself a solution. One is motivated thus to infer from the set (9.9) a formal solution having the structure

$$\tau(x,t) = \sum_{\nu=1}^{\infty} b_\nu e^{-(\nu^2\pi^2/\kappa^2 l^2)t} \sin \frac{\nu\pi x}{l}. \tag{9.10}$$

Aside from questions of convergence which are clearly to be raised in this connection, a primary matter still to be dealt with is the fulfillment of the condition (9.4) whatever the function $f(x)$ may be. Upon substituting the value $t=0$ into the formula (9.10), this is seen to devolve into the relation

$$\sum_{\nu=1}^{\infty} b_\nu \sin \frac{\nu\pi x}{l} = f(x), \qquad 0 < x < l. \tag{9.11}$$

The question of the representability of any function $f(x)$ in a series of sines is thus clearly brought into issue.

We propose to review in the following chapters Fourier's mode of coping with this problem. There are a number of reasons why a consideration of this may be regarded as worthwhile and of interest. It is, to begin with, ingenious and skillful. Aside from that it is a notable exemplar of work in the spirit of mathematics in the eighteenth century. Although this theory of Fourier's was actually created in the next century, and was crowned by the prize of the Academy of Paris in 1811, its disregard for rigor was even then outmoded and seemed, in fact, to run counter to the better standards of which Fourier himself was conscious. It is a formalism—no more—a play upon symbols in accordance with accepted rules but without much or any regard for content or significance. As such it has, of course, no place in the mathematics of our time. Fourier's work has had the profoundest effect both upon the development of pure mathematical concepts and upon the extension of the range of mathematical applications to the sciences and technology. These deserts, however, sprung in the main from Fourier's interpretations and not from his manipulations. It was, no doubt, partially because of his very disregard for rigor that he was able to take conceptual steps which were inherently impossible to men of more critical genius.

CHAPTER 10

Fourier's formal solution of his problem. [11] In the study of a representation (9.11) it comes to a mere matter of the choice of a unit of measurement to identify the length l with the value π. We shall suppose this to have been done since an appreciable formal simplification results from it. To summarize the problem at issue, then, it is that of determining from any given function $f(x)$ a set of constants b_ν such that

$$\sum_{\nu=1}^{\infty} b_\nu \sin \nu x = f(x), \tag{10.1}$$

for $0 < x < \pi$.

Fourier began his considerations of this relation by substituting in it for each sine function its power series equivalent, namely

$$\sin \nu x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\nu^{2n-1}}{(2n-1)!} x^{2n-1}.$$

Upon interchanging the order of the summations, an operation which was at that time generally resorted to without question, the relation (10.1) was made to appear in the form

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}\left(\sum_{\nu=1}^{\infty} \nu^{2n-1} b_\nu \right) x^{2n-1}.$$

The function $f(x)$ has thus been related to a series in powers of x, and since such a series must, in fact, be its MacLaurin series

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{[k]}(0) x^k, \tag{10.2}$$

in which $f^{[k]}(x)$ stands for the kth derivation of $f(x)$, it was to be concluded by a comparison of the coefficients of like powers of x that $f^{[k]}(0)$ is zero whenever k is even, and that otherwise

$$\sum_{\nu=1}^{\infty} \nu^{2n-1} b_\nu = (-1)^{n-1} f^{[2n-1]}(0), \qquad n = 1, 2, 3, \cdots. \tag{10.3}$$

The effect of this consideration has thus been to throw the constants b_ν into the rôle of the infinitely many unknowns in a system of infinitely many linear equations.

To deal with a system of this type Fourier had to invent his own method. He chose to base this upon the use of a chain of ordinary algebraic systems

$$\sum_{\nu=1}^{r} \nu^{2n-1} \beta_\nu(r) = \phi_n(r), \qquad n = 1, 2, \cdots, r, \tag{10.4}$$

of arbitrarily large degree r, the relevancy of these to the infinite system (10.3) to be assured by an adoption of the relations

$$\phi_n(\infty) = (-1)^{n-1} f^{[2n-1]}(0), \qquad n = 1, 2, 3, \cdots. \tag{10.5}$$

Inasmuch as the solution $\beta_\nu(r), \nu = 1, 2, \cdots, r$, of the system (10.4) is unique, whatever the degree r may be, it was tacitly assumed that the solution of the infinite system could be inferred from those of the finite ones in the manner

$$\beta_\nu(\infty) = b_\nu, \qquad \nu = 1, 2, 3, \cdots. \tag{10.6}$$

In the system (10.4) let the nth equation be multiplied by r^2 and let the next following equation then be subtracted from it. If this is done for each value of n from 1 to $(r-1)$ the result is the system of equations

$$\sum_{\nu=1}^{r-1} \nu^{2n-1}\{[r^2 - \nu^2]\beta_\nu(r)\} = r^2\phi_n(r) - \phi_{n+1}(r), \qquad n = 1, 2, \cdots, (r-1). \tag{10.7}$$

This suggests imposing upon the members $\phi_n(r)$ the relations

$$r^2\phi_n(r) - \phi_{n+1}(r) = \phi_n(r-1), \qquad n = 1, 2, \cdots, (r-1), \tag{10.8}$$

for, since the coefficients of the system (10.7) will then be precisely those of the system given by the equations (10.4) when r is replaced by $(r-1)$, it may be inferred that the unknowns $(r^2-\nu^2)\beta_\nu(r)$ in the one case and $\beta_\nu(r-1)$ in the other case, are, in fact, the same, namely that

$$\beta_\nu(r) = \frac{\beta_\nu(r-1)}{r^2 - \nu^2}.$$

By suitable iterations of this relation it is evidently found that

$$\beta_\nu(r) = \frac{\beta_\nu(\nu)}{\prod_{n=\nu+1}^{r} (n^2 - \nu^2)}, \qquad r > \nu, \tag{10.9}$$

whence it follows by a simple formal step from the finite r to the infinite, that

$$b_\nu = \frac{\beta_\nu(\nu)}{\prod_{n=\nu+1}^{\infty} (n^2 - \nu^2)}. \tag{10.10}$$

To any modern investigator this conclusion could, of course, be only meaningless, for the infinite product involved in it is manifestly divergent. Nor is the source of this unfortunate result difficult to trace. It lies, namely, in the naive adoption of the relations (10.8) in the face of the fact that these are quite inconsistent with the previously adopted assignments (10.5). It requires no keen critical faculty to observe at once that under the formula (10.9) the sequence of values $\beta_\nu(r)$ inevitably converges to zero with $1/r$ and hence that no conclusion other than that of the vanishing of each coefficient b_ν would be logically admissible. Fourier had no intention whatsoever of drawing that conclusion, and hence proceeded undismayed with the analysis of his formula. In so doing he found, quite naturally, that the divergence it already involved could be formally compensated for only by the introduction of still other divergencies.

Let the determinant $D(x)$ be defined by the formula

$$D(x) = \begin{vmatrix} 1 & 2 & \cdots (\nu-1) & x \\ 1^3 & 2^3 & \cdots (\nu-1)^3 & x^3 \\ 1^5 & 2^5 & \cdots (\nu-1)^5 & x^5 \\ \cdot & \cdot & \cdots\cdots & \cdot \\ 1^{2\nu-1} & 2^{2\nu-1} & \cdots (\nu-1)^{2\nu-1} & x^{2\nu-1} \end{vmatrix}, \tag{10.11}$$

and let the cofactor of its element in the ith row and jth column be denoted by $D_{i,j}$. Then since the determinant of the system (10.4) when $r=\nu$ is precisely $D(\nu)$, it is seen at once that by Cramer's rule

$$\beta_\nu(\nu) = \frac{\sum_{n=1}^{\nu} D_{n,\nu}\phi_n(\nu)}{D(\nu)}. \tag{10.12}$$

This formula can be made much more explicit. The determinant $D(x)$ is, in the first place, found to admit (*cf.* appendix V) of the evaluation

$$D(x) = (-1)^{\nu-1} D_{\nu,\nu}\, x \prod_{n=1}^{\nu-1} (n^2 - x^2). \tag{10.13}$$

Upon expanding the left-hand member of this equality by the elements of its last column, and by agreeing to define the coefficients $c_n(r)$ for any value of r not less than ν by the relations

$$\prod_{n=1,\, n\neq\nu}^{r} (n^2 - x^2) = \sum_{n=1}^{r} c_n(r)\, x^{2n-2}, \tag{10.14}$$

it evidently follows further that

$$\sum_{n=1}^{\nu} D_{n,\nu} x^{2n-1} = (-1)^{\nu-1} D_{\nu,\nu} \sum_{n=1}^{\nu} c_n(\nu)\, x^{2n-1}.$$

Now in any identical equation between polynomials or power series, the coefficients of like powers of x in the two members of the equation must be the same. It will be seen at once that because of this the substitution of any quantity whatsoever in the place of any specific power of x will not destroy the equality. In the equation above, therefore, the replacement of x^{2n-1} for each value of n by the respective quantity $\phi_n(\nu)$ is legitimate. It leads to the result

$$\sum_{n=1}^{\nu} D_{n,\nu}\phi_n(\nu) = (-1)^{\nu-1}D_{\nu,\nu}\sum_{n=1}^{\nu} c_n(\nu)\phi_n(\nu),$$

and this, together with the value of $D(\nu)$ that is obtained from the relation (10.13), yields upon substitution into the equation (10.12) the formula

$$\beta_\nu(\nu) = \frac{\sum_{n=1}^{\nu} c_n(\nu)\phi_n(\nu)}{\nu\prod_{n=1}^{\nu-1}(n^2-\nu^2)}. \tag{10.15}$$

Let it be observed now that the removal of the factor (r^2-x^2) from the left-hand member of the equation (10.14) has the mere effect of reducing the index r to $(r-1)$. From that relation it is thus seen that

$$\sum_{n=1}^{r} c_n(r)x^{2n-2} = (r^2-x^2)\sum_{n=1}^{r-1} c_n(r-1)x^{2n-2},$$

namely that

$$\sum_{n=1}^{r} c_n(r)x^{2n-2} = \sum_{n=1}^{r-1} c_n(r-1)\left[r^2x^{2n-2} - x^{2n}\right].$$

If in this each power x^{2j} is replaced by the respective value $\phi_{j+1}(r)$, it follows because of the relation (10.8) that

$$\sum_{n=1}^{r} c_n(r)\phi_n(r) = \sum_{n=1}^{r-1} c_n(r-1)\phi_n(r-1).$$

The sum on the left of this equality is thus independent of r, and this having been established it is a simple formal step to write

$$\sum_{n=1}^{\nu} c_n(\nu)\phi_n(\nu) = \sum_{n=1}^{\infty} c_n(\infty)\phi_n(\infty). \tag{10.16}$$

It was familiar in Fourier's time (*c.f.* appendix VI) that

$$\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right) = \frac{\sin \pi x}{\pi x},$$

a relation which in a formal sense, if in no other, yields the formula

$$\prod_{n=1, n\neq\nu}^{\infty}(n^2-x^2) = \left(\prod_{n=1,n\neq\nu}^{\infty} n^2\right)\left(1-\frac{x^2}{\nu^2}\right)^{-1}\frac{\sin \pi x}{\pi x}.$$

Upon replacing the left-hand member of this by the equivalent series (10.14) and substituting on the right the series equivalents

$$\left(1-\frac{x^2}{\nu^2}\right)^{-1}=\sum_{q=0}^{\infty}\frac{x^{2q}}{\nu^{2q}},$$

$$\frac{\sin \pi x}{\pi x}=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\pi^{2k-2}}{(2k-1)!}x^{2k-2},$$

it becomes

$$\sum_{n=1}^{\infty}c_n(\infty)x^{2n-2}=\left(\prod_{n=1,n\neq\nu}^{\infty}n^2\right)\sum_{q=0}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\pi^{2k-2}}{\nu^{2q}(2k-1)!}x^{2q+2k-2}.$$

The procedure of replacing each power x^{2j-2} by $\phi_j(\infty)$ leads from this to the equation

$$\sum_{n=1}^{\infty}c_n(\infty)\phi_n(\infty)=\left(\prod_{n=1,n\neq\nu}^{\infty}n^2\right)\sum_{q=0}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\pi^{2k-2}}{\nu^{2q}(2k-1)!}\phi_{q+k}(\infty),$$

or, by virtue of the relations (10.5) and (10.16), to the relation

$$\sum_{n=1}^{\nu}c_n(\nu)\phi_n(\nu)=\left(\prod_{n=1,n\neq\nu}^{\infty}n^2\right)\sum_{q=0}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{q}\pi^{2k-2}}{\nu^{2q}(2k-1)!}f^{[2q+2k-1]}(0).$$

With this result the formula (10.15) for $\beta_\nu(\nu)$ becomes wholly explicit, and through it the formula (10.10) assumes the form

$$b_\nu=\frac{\prod_{n=1,n\neq\nu}^{\infty}n^2}{\nu\prod_{n=1,n\neq\nu}^{\infty}(n^2-\nu^2)}\sum_{q=0}^{\infty}\frac{(-1)^q}{\nu^{2q}}\sum_{k=1}^{\infty}\frac{f^{[2q+2k-1]}(0)}{(2k-1)!}\pi^{2k-2}. \tag{10.17}$$

Inasmuch as all quantities in the right-hand member of this are to be regarded as known when $f(x)$ is known, this result amounts, at least in a formal sense, to an evaluation of the constant b_ν.

CHAPTER 11

The reduction and interpretation of the solution. [12] The result (10.17), although it formally accomplishes the task originally set, namely the construction of a formula through which the coefficients b_ν are expressed in terms of the given function $f(x)$, will nevertheless hardly be found completely satisfying. For the purposes of practical calculation, namely, some reduction of its intricacy would clearly be imperative, and Fourier, realizing this, turned his attention to its simplification. His effectiveness in achieving this is eloquent commentary upon his skill in analytical manipulations.

Consider, to begin with, the product of the values $(n^2-\nu^2)$ which the formula contains. The first $(\nu-1)$ factors of this are negative and have as their product

$$(-1)^{\nu-1}\prod_{n=1}^{\nu-1}(\nu-n)\prod_{n=1}^{\nu-1}(\nu+n).$$

The product of the remaining (positive) factors may be written in the form

$$\prod_{n=\nu+1}^{\infty}(n-\nu)\prod_{n=\nu+1}^{\infty}(n+\nu),$$

and thus, by simple changes of the index in each of the partial products, the entire expression may be made to appear as

$$(-1)^{\nu-1}\prod_{n=1}^{\nu-1}n\prod_{n=\nu+1}^{2\nu-1}n\prod_{n=1}^{\infty}n\prod_{n=2\nu+1}^{\infty}n.$$

Since in this each natural integer except ν and 2ν occurs twice, the excepted ones occurring just once, an alternative form for the product is evidently

$$\frac{(-1)^{\nu-1}}{2\nu^2}\prod_{n=1}^{\infty}n^2.$$

It has been found thus that formally

$$(11.1)\qquad \prod_{n=1,n\neq\nu}^{\infty}(n^2-\nu^2)=\frac{(-1)^{\nu-1}}{2}\prod_{n=1,n\neq\nu}^{\infty}n^2.$$

Consider now the formula

$$(11.2)\qquad f(x)=\sum_{n=1}^{\infty}\frac{f^{[2n-1]}(0)}{(2n-1)!}x^{2n-1},$$

which is the equivalent of the relation (10.2) by virtue of the fact that in the latter each coefficient $f^{[k]}(0)$ with an even index k is zero. A $2q$-fold term by term differentiation leads from this to the companion formula

$$f^{[2q]}(x)=\sum_{n=q+1}^{\infty}\frac{f^{[2n-1]}(0)}{(2n-2q-1)!}x^{2n-2q-1},$$

and if in this x is given the value π and the index of summation is suitably changed, the result is the equation

$$(11.3)\qquad \sum_{k=1}^{\infty}\frac{f^{[2k-2q-1]}(0)}{(2k-1)!}\pi^{2k-1}=f^{[2q]}(\pi).$$

The substitution of the evaluations (11.1) and (11.3) into the formula (10.17) causes a reduction of this latter to the form

$$(11.4)\qquad b_\nu=\frac{2}{\nu}(-1)^{\nu-1}\sum_{q=0}^{\infty}\frac{(-1)^q f^{[2q]}(\pi)}{\nu^{2q}\pi}.$$

The infinite series which still appears in this result suggests the formal definition of a function $u(x)$ by the relation

$$u(x) = \sum_{q=0}^{\infty} \frac{(-1)^q f^{[2q]}(x)}{\nu^{2q+1}}.$$

Upon differentiating this twice term by term and thereupon adjusting the index of summation, it is found that

$$u''(x) = -\nu^2 \sum_{q=1}^{\infty} \frac{(-1)^q f^{[2q]}(x)}{\nu^{2q+1}},$$

and hence that

$$u''(x) + \nu^2 u(x) = \nu f(x).$$

This is a differential equation of an elementary type. Its reduced equation is solved by the function $(c_1 \sin \nu x + c_2 \cos \nu x)$ and from this fact the method of "variation of parameters" [13] leads readily to the conclusion that the equation itself has a solution

$$U(x) = \int_0^x f(s) \sin \nu(x - s) ds.$$

This is, of course, verifiable at once by direct substitution into the differential equation. Upon setting $x = \pi$ it is thus found that

$$U(\pi) = (-1)^{\nu-1} \int_0^\pi f(s) \sin \nu s ds. \tag{11.5}$$

Let the integral in this formula now be integrated by parts $2n$ times in succession, the trigonometric factor being each time the one to be integrated. Since, as may be seen from the formula (11.2), every even ordered derivative of $f(s)$ is zero at $s = 0$, while the function $\sin \nu s$ vanishes at both $s = 0$ and $s = \pi$, the result of these integrations is the relation

$$U(\pi) = \sum_{q=0}^{n} \frac{(-1)^q f^{[2q]}(\pi)}{\nu^{2q+1}} + \frac{(-1)^{n+\nu}}{\nu^{2q+2}} \int_0^\pi f^{[2n+2]}(s) \sin \nu s ds.$$

By the step from the finite n to the infinite the formal relation

$$U(\pi) = \sum_{q=0}^{\infty} \frac{(-1)^q f^{[2q]}(\pi)}{\nu^{2q+1}},$$

may be drawn therefrom, and this together with the formula (11.5) yields the equality

$$\frac{1}{\nu} \sum_{q=0}^{\infty} \frac{(-1)^q f^{[2q]}(\pi)}{\nu^{2q}} = (-1)^{\nu-1} \int_0^\pi f(s) \sin \nu s ds.$$

By virtue of this the formula (11.4) is now once more and finally reduced to the form

$$b_\nu = \frac{2}{\pi}\int_0^\pi f(s)\sin \nu s ds. \tag{11.6}$$

This is, of course, the familiar formula in virtue of which the coefficients b_ν are generally known as those of Fourier. It is the formula (1.4) for the special case in which $l=\pi$.

Elegant though the conclusion (11.6) unquestionably is, the verdict of any critical appraisal of Fourier's accomplishment to the point of its derivation must inevitably be profoundly disappointing. As to the result, in the first place, that was not new. It had been contained in the mathematical literature for over a decade—to be precise, since the publication of the memoir of Euler that was discussed in chapter 8. While, to be sure, Euler's results applied only to functions of a certain class, that is no less true of Fourier's, since his deductions were based upon such material restrictions as the representability of the function $f(x)$ in power series of the form (11.2) that converge when $x=\pi$.

Nor could any advantage be claimed by Fourier in the matter of method. On the contrary—and even leaving aside the important fact that by its employment of divergent processes it divested itself of all rigorous validity—the method of Fourier suffers in almost every respect by comparison with that of Euler. The device of referring the problem to a system of linear equations, ingenious though it is, is nevertheless quite foreign to the nature of the problem. The trigonometric functions are conspicuously endowed with many peculiar properties and fulfill a great many characteristic interrelationships. Of this important fact Fourier's approach in no way avails itself, while Euler's, by contrast, exploits it to the utmost. In cutting directly to the heart of the matter Euler thus attained his result more perspicuously and incomparably more cheaply. In this respect the superiority is all his.

With the priority and preference in manipulative matters thus denied him, Fourier's claim to renown must be based upon other grounds, and these are, namely, those of interpretation. Approaching the formula (11.6) afresh, without regard for the manner of its derivation, it was observed by him that through it each coefficient b_ν admits of interpretation as the area between the abscissas $x=0$ and $x=\pi$, and under the graph

$$y = \frac{2}{\pi} f(x)\sin \nu x. \tag{11.7}$$

Such an area is evidently conceivable, and retains its clear-cut significance, in association with functions $f(x)$ that are in a very general sense quite arbitrary. Certainly these functions need not be assumed to be continuous or expansible by any simple analytical formulas. They might be graphically defined and might well represent distributions of functional values that are extremely erratic. On the basis of such considerations Fourier concluded that *any* and *every* function $f(x)$ had associated with it a set of constants b_ν.

From this fact alone it would not follow, of course, that with such coefficients the representation of the function $f(x)$ by a series (10.1) would result. As has been seen in chapters 1 and 5, the masters of the eighteenth century had rejected such a possibility as manifestly absurd. In this matter, however, Fourier was willing to disregard opinions and precedents, however well established, and to look further for himself. From calculations of the coefficients b_ν with small indices ν in the cases of a great variety of functions $f(x)$, and from subsequent plottings of the respective initial segments of the resulting trigonometric series, he came to convictions upon two salient points, namely: (i) that the series (10.1) *always* represents the function over the interval $0<x<\pi$, and (ii) that in general this representation does not persist for values of x outside that interval.

Fourier's announcement of these facts was quite generally met with incredulity. Even the mass of his substantiating evidence won, in many cases, only grudging and reluctant acceptance. The implications behind the new assertions were too revolutionary to be easily assimilated. They called for no less than a fundamental revision of many concepts that were wholly traditional, some of them lying at the very basis of mathematical analysis. On the other hand Fourier's new theory did now finally vindicate the half century old reasoning of Daniel Bernoulli by which he had convinced himself, if no others, that any curve from which a taut elastic string could spring into vibration could be represented by a trigonometric series.

CHAPTER 12

The Dirichlet integrals. Once Fourier had deduced the formula (11.6), he, like Euler before him, observed that in a schematic way the result is recoverable in a most direct and simple manner by the mere expedient of multiplying the relation (10.1) through by $\sin \mu x$ with any natural integer μ, and then integrating term by term over the interval $(0, \pi)$. The infinite series reduces under this process to a single term, because of the evaluations

$$(12.1) \qquad \int_0^\pi \sin \nu x \sin \mu x dx = 0, \qquad \text{for } \nu \neq \mu,$$

and the formula for b_μ thus emerges. The procedure is obviously adaptable also to the case of a cosine representation

$$(12.2) \qquad f(x) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} a_\nu \cos \nu x,$$

the evaluations

$$\int_0^\pi \cos \nu x \cos \mu x dx = 0, \qquad \text{for } \nu \neq \mu, \tag{12.3}$$

leading in this case to the formulas

$$a_\nu = \frac{2}{\pi} \int_0^\pi f(s) \cos \nu s ds, \qquad \nu = 1, 2, 3, \cdots . \tag{12.4}$$

An extension of these results to the case of a function that is given over the larger interval $(-\pi, \pi)$ is of importance and is easily deduced. Through the relation

$$f(x) = f_0(x) + f_e(x),$$

with

$$f_0(x) \equiv \tfrac{1}{2}[f(x) - f(-x)],$$
$$f_e(x) \equiv \tfrac{1}{2}[f(x) + f(-x)],$$

the function $f(x)$ is expressed as the sum of two components of which the first one is an odd function and the second one even. Now the relations (10.1) and (12.2) for these respective functions, namely

$$f_0(x) = \sum_{\nu=1}^{\infty} b_\nu \sin \nu x,$$

$$f_e(x) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} a_\nu \cos \nu x,$$

obviously remain quite unchanged if x is replaced by $-x$. Any validity they may have over the interval $(0, \pi)$ therefore implies the same over the larger interval $(-\pi, \pi)$. It follows at once that for the originally given function the representation in question is of the form

$$f(x) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} [a_\nu \cos \nu x + b_\nu \sin \nu x]. \tag{12.5}$$

The formulas for the coefficients in this are, moreover, easily freed from reference to the components $f_0(x)$ and $f_e(x)$, and may thus be brought to expression directly in terms of the function $f(x)$ itself. Since

$$\int_0^\pi f_0(s) \sin \nu s ds = \frac{1}{2} \int_{-\pi}^\pi f(s) \sin \nu s ds,$$

as may easily be verified, and also

$$\int_0^\pi f_e(s) \cos \nu s ds = \frac{1}{2} \int_{-\pi}^\pi f(s) \cos \nu s ds,$$

the formulas in question are, namely, seen to be

$$(12.6)\qquad \begin{aligned} a_\nu &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\cos \nu s ds,\\ b_\nu &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\sin \nu s ds. \end{aligned}$$

In the instance that the interval $(-\pi, \pi)$ which has been taken to be basic is replaced by the more general interval $(-l, l)$, the representation (12.5), (12.6), is correspondingly replaced by that given in the formulas (1.1), (1.2).

Let the sum of the first $2N+1$ terms of the series (12.5) be designated by $S_N(x)$, thus

$$(12.7)\qquad S_N(x) = \frac{a_0}{2} + \sum_{\nu=1}^{N} [a_\nu \cos \nu x + b_\nu \sin \nu x].$$

The substitution of the values of the coefficients (12.6) into this gives it the aspect

$$S_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\left[\frac{1}{2} + \sum_{\nu=1}^{N} \cos \nu(s-x) ds\right],$$

and this may be contracted by the use of elementary trigonometric relations (*cf.* appendix VII) into the wholly compact form

$$(12.8)\qquad S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\,\frac{\sin\,[(N+\frac{1}{2})(s-x)]}{\sin\,[\frac{1}{2}(s-x)]}\,ds.$$

An equivalent manner of writing this, and one that has some analytic advantages, is

$$(12.9)\quad S_N(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\,\frac{\sin\,[(N+\frac{1}{2})(s-x)]}{(s-x)}\,ds + \int_{-\pi}^{\pi} F(s)\Psi(s, x, N)ds,$$

with the symbols $F(s)$ and $\Psi(s, x, N)$ having the significance

$$F(s) \equiv \frac{f(s)}{2\pi}\left\{\frac{1}{\sin\,[\frac{1}{2}(s-x)]} - \frac{1}{\frac{1}{2}(s-x)}\right\},$$

$$\Psi(s, x, N) \equiv \sin\,[(N+\tfrac{1}{2})(s-x)].$$

From the latter of these formulas it is easily inferred that

$$|\Psi(s, x, N)| \leqq 1, \quad \text{and}$$

$$\left|\int_{\alpha}^{\beta} \Psi(s, x, N)ds\right| \leqq \frac{4}{2N+1}, \quad \text{for } -\pi \leqq \alpha < \beta \leqq \pi.$$

The function $\Psi(s, x, N)$ thus possesses the properties:

(i) that it is bounded uniformly as to N and s;

(ii) that its integral over any sub-interval of the range $(-\pi, \pi)$ converges to zero with $1/N$ uniformly as to the sub-interval.

These properties are sufficient to insure [14] the relation

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} F(s)\Psi(s, x, N)ds = 0, \tag{12.10}$$

whenever the function $F(s)$ is integrable (in the sense of Lebesgue) over the interval $(-\pi, \pi)$.

Now from the definition of the function $F(s)$ it may readily be seen that integrability is assured to it by that of the function $f(s)$ provided the point x is in the interior of the interval $(-\pi, \pi)$. Thus for every integrable function $f(x)$ the final integral in the relation (12.9) converges to zero, and that relation thus implies that

$$\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) \frac{\sin\left[(N + \frac{1}{2})(s - x)\right]}{(s - x)} ds, \tag{12.11}$$

whenever the right-hand limit involved in this exists. This permits us at once the conclusion: that any function $f(x)$ which is integrable and which is furthermore such that for it

$$\lim_{N\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} f(s) \frac{\sin\left[(N + \frac{1}{2})(s - x)\right]}{(s - x)} ds = f(x), \tag{12.12}$$

is a function which is representable by a trigonometric series in the manner (12.5), (12.6).

Although Fourier made calculations upon many specific cases, he gave no general proof of his ultimate assertion of the trigonometric representability of *arbitrary* functions. Indeed no such proof could have been given, since in the omission of all qualifications upon the functions the assertion is too broad to be valid. The first proof that was both satisfactory in the matter of rigor and ample in the matter of generality was given by Dirichlet in the year 1829. In the manner that has been indicated above, this proof was based by Dirichlet upon an establishment of the relation (12.12). The integrals involved in that relation and in (12.8) are accordingly known generally as "Dirichlet integrals." Dirichlet's proof in its original form, or as it has been improved and refined, is to be found at many places in the mathematical literature [15]. We shall, therefore, go no further into it here but shall draw this part of the discussion to its close. In the following part the primary subject of study is to be a generalization of the entire theory in which the representations of functions in trigonometric terms sink to the status of special cases.

PART II

CHAPTER 13

The differential boundary problem. Among the most powerful of mathematical means for the formulation of natural laws are the linear differential equations of either the partial or the ordinary type. Varied and diverse as physical phenomena certainly are, they nevertheless submit quite generally to description by such equations. The flow of heat in material bodies and the vibrations of elastic strings under tension are instances of this that have already been noted.

By its very nature as the formulation of a more or less general law, any particular differential equation applies, of course, to the entire category of manifestations which the law itself governs. To single out for description any individual phenomenon from such a category, therefore requires that some auxiliary means beyond the equation itself be resorted to. This ordinarily takes on the form of a set of one or more restricting relations that are expressive of the characterizing initial or boundary conditions. The differential equation together with such relations is commonly designated as a *differential system*. It is also said to define a *differential boundary problem*. The system consisting of the equations (4.6) and (4.7), for instance, thus defines a partial differential boundary problem, namely the one which is descriptive of a certain stretched string vibrating with specified end points and initial position of release. The equations (9.2), (9.3) and (9.4), define a similar problem, one which describes a linear flow of heat from specific initial temperatures and under certain boundary conditions.

The method that was employed in the reduction of the boundary problem of chapter 9 was designated there as that of the "separation of variables." Its immediate effect was to refer the partial differential system to an *ordinary* system (9.6), this latter having the peculiarity of involving an unspecified constant or parameter, which was designated by λ_ν. Following the solution of this ordinary boundary problem at characteristic values of this parameter, the theory led in a natural way to the further problem of representing an arbitrary function in terms of the respective solutions. This method was in no way especially designed for the problem of chapter 9. It is on the contrary one that is highly flexible and of very wide applicability. In the particular instance there considered the ordinary differential equation which characterized the problem was one whose solutions were trigonometric functions, and it was because of that, that the representation of the function $f(x)$ took the form (9.11), namely that of the Fourier theory. This feature was special, to the extent that it would not even have maintained for the equation (9.2) if the coefficient κ^2 involved in it had been dependent upon the coördinate x, rather than constant. In the following a discussion is to be framed which is free from such peculiar specializations.

Let the variable x be real, with the range

$$a \leqq x \leqq b, \tag{13.1}$$

and on this interval let $p(x)$, $q(x)$, and $r(x)$, be differentiable functions. The symbol $L(\phi, \lambda)$ is to designate the differential expression

$$L(\phi, \lambda) \equiv \phi'' + p(x)\phi' + [q(x)\lambda + r(x)]\phi. \tag{13.2}$$

In this λ is to play the rôle of a parameter, the range of which is to be the entire complex plane. The differential equation

$$L(\phi, \lambda) = 0, \tag{13.3}$$

is, then, one that is regular, in the sense that it has no singular points upon the interval (13.1). As an equation of the second order it will, of course, not generally be explicitly solvable. Certain facts concerning its solutions are, however, familiar. Of these the following ones will be especially relevant to the discussion proposed [16].

(i) The equation admits of solutions $\phi(x, \lambda)$ that have continuous second derivatives as to x, and that are analytic in λ over the entire complex λ plane.

(ii) There is a pair of such solutions $\phi_1(x, \lambda)$, $\phi_2(x, \lambda)$, that are linearly independent as functions of x for all values of λ.

(iii) The Wronskian $\Omega(x, \lambda)$ of this pair, namely the determinant

$$\Omega(x, \lambda) \equiv \begin{vmatrix} \phi_1(x, \lambda) & \phi_2(x, \lambda) \\ \phi_1'(x, \lambda) & \phi_2'(x, \lambda) \end{vmatrix}, \tag{13.4}$$

is subject to the relation

$$\Omega(x, \lambda) = \Omega(a, \lambda)e^{-\int_a^x p(x)\,dx}, \tag{13.5}$$

with $\Omega(a, \lambda)$ an analytic function of λ that is different from zero for all λ.

(iv) The general solution of the equation (13.3) is expressible in terms of the pair $\phi_1(x, \lambda)$, $\phi_2(x, \lambda)$ by a form

$$h_2\phi_1(x, \lambda) + h_1\phi_2(x, \lambda), \tag{13.6}$$

in which the coefficients h_1, h_2 are constants as to x, though they may be functions of λ.

For any equation (13.3) there are known to be infinitely many pairs of solutions $\phi_1(x, \lambda)$, $\phi_2(x, \lambda)$, that have the properties enumerated. Of these pairs any one serves in every way as well as any other, and the choice that is made is accordingly immaterial. For the sake of avoiding gratuitous complications, however, it will be supposed throughout the discussion that when a choice of such a pair in the instance of any specific equation has been made, it will be consistently adhered to. To that extent, then, the designations $\phi_1(x, \lambda)$, $\phi_2(x, \lambda)$ will be understood to apply not to random but to specific solutions.

Let the symbols $\beta_{i,j}, \gamma_{i,j}, i = 1, 2; j = 1, 2, 3, 4$; now denote any constants which are such that when

$$\alpha_{i,j}(\lambda) = \beta_{i,j}\lambda + \gamma_{i,j}, \tag{13.7}$$

then the matrix

$$\begin{Vmatrix} \alpha_{1,1}(\lambda) & \alpha_{1,2}(\lambda) & \alpha_{1,3}(\lambda) & \alpha_{1,4}(\lambda) \\ \alpha_{2,1}(\lambda) & \alpha_{2,2}(\lambda) & \alpha_{2,3}(\lambda) & \alpha_{2,4}(\lambda) \end{Vmatrix} \tag{13.8}$$

is of the rank 2 for every value of λ. In terms of the values of any function θ and its derivative θ' at $x=a$ and $x=b$, the forms

$$A_i(\theta, \lambda) \equiv \alpha_{i,1}(\lambda)\theta'(a) + \alpha_{i,2}(\lambda)\theta(a) + \alpha_{i,3}(\lambda)\theta'(b) + \alpha_{i,4}(\lambda)\theta(b), \quad i = 1,2, \tag{13.9}$$

are then always linearly independent. Under these circumstances the differential system

$$\begin{aligned} L(u, \lambda) &= 0, \\ A_1(u, \lambda) &= 0, \\ A_2(u, \lambda) &= 0, \end{aligned} \tag{13.10}$$

defines an ordinary boundary problem. It is this problem which is to be central to our discussion. It will be seen at once to include as a special case the problem (9.6), and to do that even if in the latter the coefficient κ^2 varies with x. Many other boundary problems that stem from physical origins are also included, as will upon occasion be seen in the following.

CHAPTER 14

The characteristic values and solutions. Since any solution $u(x)$ of the differential system (13.10) must in particular solve the differential equation (13.3), it must have the form

$$u(x) = h_2\phi_1(x, \lambda) + h_1\phi_2(x, \lambda). \tag{14.1}$$

Except for the trivial solution $u(x)\equiv 0$, which we shall herewith specifically and permanently rule out of this discussion, the values h_1, h_2 will not both be zero. The substitution of this form into the boundary relations of the system give to the latter the aspect

$$\begin{aligned} h_2A_{1,1}(\lambda) + h_1A_{1,2}(\lambda) &= 0, \\ h_2A_{2,1}(\lambda) + h_1A_{2,2}(\lambda) &= 0, \end{aligned} \tag{14.2}$$

in which the abbreviations

$$A_{i,j}(\lambda) \equiv A_i(\phi_j(x, \lambda), \lambda), \qquad i, j = 1, 2, \tag{14.3}$$

have been resorted to.

The equations (14.2) constitute an algebraic system in which the values h_1, h_2, function as the unknowns. Since this system is homogeneous its nontrivial solvability is contingent upon the vanishing of its determinant $\Delta(\lambda)$, where

$$(14.4) \qquad \Delta(\lambda) \equiv \begin{vmatrix} A_{1,1}(\lambda) & A_{1,2}(\lambda) \\ A_{2,1}(\lambda) & A_{2,2}(\lambda) \end{vmatrix}.$$

A proper solution of the boundary problem thus exists if and only if λ is a root of the so-called *characteristic equation*

$$(14.5) \qquad \Delta(\lambda) = 0.$$

These roots are called the *characteristic values* (*Eigenwerte*) of the boundary problem. The multiplicity with which such a value occurs as a root of the equation (14.5) is also designated to be its *multiplicity* as a characteristic value. If the root is one at which the elements of the determinant (14.4) do not all vanish, namely at which the rank of the determinant is 1, it is said to be a characteristic value of the *index* 1. On the other hand a value at which the elements do all vanish is said to be of the index 2. It is not difficult to see that in this latter case its multiplicity must also be at least 2, and hence that in every case

$$(14.6) \qquad \text{(The index)} \leqq \text{(The multiplicity)}.$$

There is material advantage to be gained by formally regarding a value whose index is 2 as being, in fact, two coincident characteristic values. We shall herewith, once for all, adopt this convention.

The determinant $\Delta(\lambda)$ is an analytic function for all values of λ. The number of its zeros in any finite region of the λ plane is, therefore, finite. Thus, in particular, only a finite number of these zeros fulfill a relation $|\lambda| < N$, whatever the constant N may be, a fact from which it follows that they—the characteristic values—may be sequentially ordered in a succession of non-decreasing absolute value. With the assignment of subscripts in such a succession, the characteristic values thus follow each other in the array

$$(14.7) \qquad \lambda_0, \lambda_1, \lambda_2, \lambda_3, \cdots .$$

In this, then,

$$(14.8) \qquad |\lambda_r| \leqq |\lambda_{r+1}|,$$

for every subscript r. Each characteristic value of the index 2 occurs in the array twice, occupying, as we may and shall assume, two consecutive positions. Each characteristic value of the index 1 occurs just once.

Consider now any characteristic value λ_r that is of the index 1. The pairs of values $(h_1, h_2) = (h_1^{(r)}, h_2^{(r)})$ which satisfy the system (14.2), namely for which

$$(14.9) \qquad \begin{aligned} h_2^{(r)} A_{1,1}(\lambda_r) + h_1^{(r)} A_{1,2}(\lambda_r) &= 0, \\ h_2^{(r)} A_{2,1}(\lambda_r) + h_1^{(r)} A_{2,2}(\lambda_r) &= 0, \end{aligned}$$

have in this case members that stand in a fixed ratio to each other. Any such pair yields through the formula

$$(14.10) \qquad u_r(x) = h_2^{(r)} \phi_1(x, \lambda_r) + h_1^{(r)} \phi_2(x, \lambda_r),$$

an associated solution for the boundary problem, and the solution so obtainable from any other pair is a mere multiple of this. The function $u_r(x)$ is known as a *characteristic solution* (*Eigenfunktion*) of the boundary problem.

If now, on the other hand, λ_r and λ_{r+1} are the designations of a characteristic value of the index 2, then at this value the system (14.9) is vacuous since each quantity $A_{i,j}(\lambda_r)$ is zero. There is, therefore, no restriction upon the choice of h_1, h_2, and hence in particular two pairs that are not multiples of each other may be taken. These lead through the formula (14.1) to two linearly independent characteristic solutions $u_r(x)$, $u_{r+1}(x)$, which we may regard as associated with the values λ_r and λ_{r+1} respectively. Thus in every instance each symbol λ_r has associated with it a function $u_r(x)$.

There exist boundary problems that have only a finite number of characteristic values or even none at all. There also exists, however, a large class of such problems for which the characteristic values are infinitely numerous. It is only to problems of this latter category that the continuing discussion will, in all its phases, be relevant.

CHAPTER 15

The adjoint boundary problem. With the coefficient functions $p(x)$, $q(x)$, and $r(x)$, that occur in the expression (13.2) let $M(\phi, \lambda)$ be defined by the formula

$$M(\phi, \lambda) \equiv \phi'' - (p\phi)' + [q(x)\lambda + r(x)]\phi. \tag{15.1}$$

This differential form is said to be *adjoint to* the form $L(\phi, \lambda)$. It is customary also to refer to it as *the adjoint of* $L(\phi, \lambda)$. If it is completely written out, thus

$$\phi'' - p(x)\phi' + [q(x)\lambda + r(x) - p'(x)]\phi,$$

and its adjoint is in turn constructed, this latter is found to be again the form $L(\phi, \lambda)$. The relationship of being adjoint is thus a reciprocal one, either of two forms so associated being the adjoint of the other. For the two adjoint forms the equality

$$\psi L(\phi, \lambda) - \phi M(\psi, \lambda) = \frac{d}{dx} Q(\phi, \psi, x), \tag{15.2}$$

with

$$Q(\phi, \psi, x) \equiv \phi'(x)\psi(x) - \phi(x)\psi'(x) + p(x)\phi(x)\psi(x), \tag{15.3}$$

is easily verified for any two suitably differentiable functions ϕ and ψ. The relation is thus an identity. It is generally known as the "Lagrange identity," and is the source of many important analytical formulas.

The notion of the adjoint relationship is extensible in the most direct and immediate way to differential equations such as the equation (13.3). This latter and the equation

$$M(\psi, \lambda) = 0, \tag{15.4}$$

are thus likewise said to be adjoint. As the discussion proceeds there will be ample illustration of the manner in which the differential equations of an adjoint pair, or their respective solutions, interplay in the development of a theory. Even here it may be observed that the solubility of either equation implies that of the other, since the solutions of either are simply expressible in terms of those of the other. Thus if the solutions $\phi_1(x, \lambda)$, $\phi_2(x, \lambda)$ of the equation (13.3) and their Wronskian (13.4) are used to construct the functions $\psi_1(x, \lambda)$, $\psi_2(x, \lambda)$ by the formulas

$$\psi_1(x, \lambda) \equiv \frac{-\phi_2(x, \lambda)}{\Omega(x, \lambda)}, \qquad \psi_2(x, \lambda) \equiv \frac{\phi_1(x, \lambda)}{\Omega(x, \lambda)}, \tag{15.5}$$

these latter are found by direct substitution to satisfy the equation (15.4). They are similarly seen to fulfill the relations

$$Q(\phi_i, \psi_j, x) = \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{if } j \neq i, \end{cases} \tag{15.6}$$

and to be linearly independent. In terms of them the general solution of the equation (15.4) is accordingly expressible in the form

$$v(x, \lambda) = k_2\psi_1(x, \lambda) + k_1\psi_2(x, \lambda), \tag{15.7}$$

with coefficients k_1, k_2 that are free from x. The relations (15.6) when applied to the form (15.7) yield at once the evaluations

$$\begin{aligned} Q(\phi_1, v, x) &= k_2, \\ Q(\phi_2, v, x) &= k_1. \end{aligned} \tag{15.8}$$

The extension of the notion of the adjoint relationship to the complete boundary problem (13.10), although it is not immediate is nevertheless possible, and may in fact be made in several formally different but essentially equivalent ways. We shall do this in the following manner. Consider the differential system

$$\begin{aligned} M(v, \lambda) &= 0, \\ v(a) &= \mu_2\alpha_{1,1}(\lambda) + \mu_1\alpha_{2,1}(\lambda), \\ -v'(a) + p(a)v(a) &= \mu_2\alpha_{1,2}(\lambda) + \mu_1\alpha_{2,2}(\lambda), \\ -v(b) &= \mu_2\alpha_{1,3}(\lambda) + \mu_1\alpha_{2,3}(\lambda), \\ v'(b) - p(b)v(b) &= \mu_2\alpha_{1,4}(\lambda) + \mu_1\alpha_{2,4}(\lambda), \end{aligned} \tag{15.9}$$

in which there occurs besides the function $v(x)$ also a pair of "parameters" μ_1, μ_2 that are independent of x. The coefficients $\alpha_{i,j}(\lambda)$ are to be those which were defined by the formulas (13.7). We shall show that this system in fact defines a boundary problem which is essentially of the type (13.10), and that the parameters μ_1, μ_2 may be looked upon as standing for certain specific linear forms in the values $v'(a)$, $v(a)$, $v'(b)$ and $v(b)$, with coefficients that are functions of λ.

The condition that the set of boundary relations of the system (15.9) be consistent in the "unknowns" μ_1, μ_2, is that the matrix

$$\left\| \begin{array}{rcc} v(a) & \alpha_{1,1} & \alpha_{2,1} \\ -v'(a)+p(a)v(a) & \alpha_{1,2} & \alpha_{2,2} \\ -v(b) & \alpha_{1,3} & \alpha_{2,3} \\ v'(b)-p(b)v(b) & \alpha_{1,4} & \alpha_{2,4} \end{array} \right\|$$

be of the rank 2. Now by hypothesis there is, for every λ, some two rowed determinant from the last two columns of this matrix that is not zero. This occurs as a minor in two of the three rowed determinants of the matrix. The results of setting these latter equal to zero are two equations in the quantities $v'(a)$, $v(a)$, $v'(b)$, $v(b)$. That these equations are independent follows at once from the fact that each contains one of the four quantities which the other does not contain. Thus the differential equation of the system (15.9) is seen to have imposed upon it two linear boundary conditions. A boundary problem is thus defined. The boundary relations of this problem evidently have coefficients that are polynomials in λ. To this significant extent the problem is accordingly similar in form to the problem (13.10). It is true that the coefficients of the boundary relations of the system (13.10) were taken to be linear polynomials, whereas those of the newly found system may be quadratic. That, however, is not truly important, for the assumption of the coefficients of the problem (13.10) to be of the first degree in λ was motivated only by the desire for simplicity, and is in no way essential. Finally some pair of the equations (15.9) is always solvable for μ_1 and μ_2. By that solution μ_1 and μ_2 are expressed as linear forms in the values $v'(a)$, $v(a)$, $v'(b)$, $v(b)$ as was asserted above to be possible.

It is a matter to be observed that if in the equations (15.9) the parameters μ_1, μ_2 were to be both zero, it would follow that $v'(a)$ and $v(a)$ would also necessarily vanish. Only the trivial solution of the differential equation $M(v, \lambda)$ conforms to these values. Since this solution is to be ruled out of the discussion, it is evident that the simultaneous vanishing of μ_1 and μ_2 is likewise to be barred. It is to be understood henceforth, therefore, that of the value pair μ_1, μ_2 at least one member is in every case different from zero.

If in any differential system the differential equation is replaced by its general solution, and the boundary relations are replaced by independent linear combinations of them, the content of the system clearly remains unchanged. In consonance with this let the first two of the boundary relations of the set (15.9) be multiplied respectively by the factors $\phi_j'(a, \lambda)$ and $\phi_j(a, \lambda)$ with $j=1, 2$, and let them then be added. The left-hand members thus obtained are found to be $Q(\phi_j, v, a)$, and thus by the relations (15.8) the resulting equalities are

$$k_{3-j} = \mu_2 \overset{(a)}{A}_{1,j}(\lambda) + \mu_1 \overset{(a)}{A}_{2,j}(\lambda), \qquad j = 1, 2, \tag{15.10}$$

with

$$A_{i,j}^{(a)}(\lambda) \equiv \alpha_{i,1}(\lambda)\phi_j'(a, \lambda) + \alpha_{i,2}(\lambda)\phi_j(a, \lambda), \qquad i, j = 1, 2. \tag{15.11}$$

In a similar way the last two of the relations (15.9) may be combined with the multipliers $-\phi_j'(b, \lambda)$ and $-\phi_j(b, \lambda)$ to assume the forms

$$k_{3-j} = \mu_2[A_{1,j}^{(a)}(\lambda) - A_{1,j}(\lambda)] + \mu_1[A_{2,j}^{(a)}(\lambda) - A_{2,j}(\lambda)], \qquad j = 1, 2. \tag{15.12}$$

In content the equations (15.7), (15.10) and (15.12) are thus equivalent to the set (15.9).

Consider now the case in which λ is any characteristic value of the boundary problem (15.9), namely a value for which a set of elements $v(x)$, μ_1, μ_2, fulfilling the equations (15.9) exists. From the fulfillment of the equations (15.10) and (15.12) it follows then at once that

$$\begin{aligned} \mu_2 A_{1,1}(\lambda) + \mu_1 A_{2,1}(\lambda) &= 0, \\ \mu_2 A_{1,2}(\lambda) + \mu_1 A_{2,2}(\lambda) &= 0. \end{aligned} \tag{15.13}$$

Since μ_1 and μ_2 are not both zero, the determinant of this system must vanish. This determinant is, however, precisely $\Delta(\lambda)$, as that was defined by the formula (14.4). Thus λ must be a root of the equation (14.5), namely a characteristic value of the boundary problem (13.10). Every characteristic value of the adjoint boundary problem is thus also a characteristic value of the given one.

The converse of this may also be established. Thus let λ_n be any value from the set (14.7). With λ at this value the system (15.13) is non-trivially satisfiable by values $\mu_1^{(n)}$, $\mu_2^{(n)}$, which, of course, fulfill the relations

$$\begin{aligned} \mu_2^{(n)} A_{1,1}(\lambda_n) + \mu_1^{(n)} A_{2,1}(\lambda_n) &= 0, \\ \mu_2^{(n)} A_{1,2}(\lambda_n) + \mu_1^{(n)} A_{2,2}(\lambda_n) &= 0. \end{aligned} \tag{15.14}$$

Let $v_n(x)$ be the solution of the first three equations of the set (15.9) with $\lambda = \lambda_n$ and $\mu_j = \mu_j^{(n)}$, $j = 1, 2$. The familiar "existence theorem" for a linear ordinary differential equation [17] gives assurance of both the existence and the uniqueness of this function. Now with the values at hand the equations (15.10) are fulfilled. Because of this, however, and with the equations (15.14), the relations (15.12) are seen to be also fulfilled. This means that the entire system (15.9) admits this solution, namely that λ_n is also a characteristic value of the boundary problem (15.9). We may go even somewhat further. If the index of λ_n relative to the boundary problem (13.10) is 1, the system (15.14) determines the values $\mu_1^{(n)}$, $\mu_2^{(n)}$, except for a common multiplicative factor which remains arbitrary. From the manner in which $v_n(x)$ is determined it is then seen that this function is also fixed except for a constant multiplier. If, on the other hand, λ_n is of the index 2 relative to the boundary problem (13.10), the system (15.14) admits of solution by two linearly independent pairs of values μ_1, μ_2. Each of these leads in the manner described to an associated solution $v(x)$, and the two of these so obtained are also linearly independent. It has thus been shown that adjoint boundary problems have the same characteristic values, and, moreover, that each such value has the same index relative to each of the two problems.

For the characteristic solutions $v_n(x)$ of the boundary problem (15.9) the formula (15.7) yields the form

$$(15.15) \qquad v_n(x) = k_2^{(n)}\psi_1(x, \lambda_n) + k_1^{(n)}\psi_2(x, \lambda_n).$$

The coefficients in this, as they may be drawn from the equations (15.10), have the evaluations

$$(15.16) \qquad \begin{aligned} k_2^{(n)} &= \mu_2^{(n)} A_{1,1}^{(a)}(\lambda_n) + \mu_1^{(n)} A_{2\,1}^{(a)}(\lambda_n), \\ k_1^{(n)} &= \mu_2^{(n)} A_{1,2}^{(a)}(\lambda_n) + \mu_1^{(n)} A_{2,2}^{(a)}(\lambda_n). \end{aligned}$$

In general the boundary problems (13.10) and (15.9) are distinct. In a restricted class of cases, however, they may be effectively the same. Boundary problems of this class are said to be *self-adjoint*. Many familiar physical phenomena admit of mathematical formulations in terms of self-adjoint differential systems.

CHAPTER 16

Generalized orthogonality. Of the methods for the determination of coefficients in a trigonometric representation, that of Euler (Chapter 8) was seen to be by all odds the simpler one. It is more direct and very much shorter than Fourier's (Chapters 10, 11). And this advantage was recognized in Chapter 11 to be attributable essentially to the fact that from the very start it exploits strategically the peculiar properties of the functions in terms of which the representation is made. These properties are particularly those which enter into the so-called *orthogonality* of the trigonometric functions, namely, those which come to their most familiar expression, at least in part, in the relations (12.1), (12.3), (8.10) *etc.* A directive influence upon the present discussion is evidently to be discerned in this fact. The rôle that is filled by the trigonometric functions in the classical theory is to be assigned in this generalization to the characteristic solutions of the boundary problem (13.10). The discovery of interrelations between these solutions, and especially of such as reduce to orthogonality under suitable specialization, therefore appears as a significant issue at this turn.

In terms of the coefficients $h_1^{(r)}$, $h_2^{(r)}$, with which the relation (14.10) maintains, let the function $U_r(x, \lambda)$ be defined by the formula

$$(16.1) \qquad U_r(x, \lambda) \equiv h_2^{(r)}\phi_1(x, \lambda) + h_1^{(r)}\phi_2(x, \lambda).$$

This is evidently a solution of the differential equation (13.3), specifically one which becomes a characteristic solution of the boundary problem (13.10) when λ is given the value λ_r. If, then, as usual, $v_n(x)$ is used to designate the nth characteristic solution of the adjoint problem, it is clear that

$$L(U_r, \lambda) = 0,$$
$$M(v_n, \lambda) = (\lambda - \lambda_n)q(x)v_n(x).$$

In virtue of these relations the identity (15.2), with $U_r(x, \lambda)$ and $v_n(x)$ in the place of ϕ and ψ, yields, upon integration, the equation

$$(16.2) \qquad (\lambda - \lambda_n)\int_a^b q(x)U_r(x, \lambda)v_n(x)dx = Q(U_r, v_n, a) - Q(U_r, v_n, b).$$

Now when $\mu_1^{(n)}$, $\mu_2^{(n)}$ and λ_n stand in the place of μ_1, μ_2 and λ, the boundary relations of the system (15.9) are fulfilled by the function $v_n(x)$, while their right-hand members may be written as

$$\mu_2^{(n)}[\alpha_{1,j}(\lambda) - (\lambda - \lambda_n)\beta_{1,j}] + \mu_1^{(n)}[\alpha_{2,j}(\lambda) - (\lambda - \lambda_n)\beta_{2,j}], \qquad j = 1, 2, 3, 4.$$

Upon multiplying them respectively by $U_r'(a, \lambda)$, $U_r(a, \lambda)$, $U_r'(b, \lambda)$, $U_r(b, \lambda)$, and then adding them, it is accordingly found that

$$\begin{aligned} Q(U_r, v_n, a) - Q(U_r, v_n, b) &= \mu_2^{(n)}A_1(U_r, \lambda) + \mu_1^{(n)}A_2(U_r, \lambda) \\ &\quad - (\lambda - \lambda_n)[\mu_2^{(n)}B_1(U_r) + \mu_1^{(n)}B_2(U_r)], \end{aligned}$$

the symbols B_i having been introduced here in the sense

$$(16.3) \qquad B_i(\phi) \equiv \beta_{i,1}\phi'(a) + \beta_{i,2}\phi(a) + \beta_{i,3}\phi'(b) + \beta_{i,4}\phi(b), \qquad i = 1, 2.$$

The equation (16.2) therefore assumes, after a division by $(\lambda - \lambda_n)$, the form

$$(16.4) \qquad \int_a^b q(x)U_r(x, \lambda)v_n(x)dx + \mu_2^{(n)}B_1(U_r) + \mu_1^{(n)}B_2(U_r) = \Phi_{n,r}(\lambda),$$

in which the right-hand member is explicitly given by the formula

$$(16.5) \qquad \Phi_{n,r}(\lambda) \equiv \frac{\mu_2^{(n)}A_1(U_r, \lambda) + \mu_1^{(n)}A_2(U_r, \lambda)}{\lambda - \lambda_n}, \qquad \text{if } \lambda \neq \lambda_n.$$

We are to be specifically concerned with the form of the relation (16.4) and hence with the value of the expression (16.5) when $\lambda = \lambda_r$.

From the relations (16.1) it is to be seen at once that

$$(16.6) \qquad A_j(U_r, \lambda) = h_2^{(r)}A_{j,1}(\lambda) + h_1^{(r)}A_{j,2}(\lambda), \qquad j = 1, 2.$$

The equations (14.9) thus assure the relations

$$(16.7) \qquad A_j(U_r, \lambda_r) = 0, \qquad j = 1, 2,$$

and from these it follows that the expression (16.5) vanishes at λ_r, whenever $\lambda_r \neq \lambda_n$. For its evaluation at $\lambda = \lambda_n$ we may observe that by virtue of the formulas (16.6) the numerator of the expression (16.5) may be written out explicitly as

$$h_2^{(r)}\big[\mu_2^{(n)}A_{1,1}(\lambda) + \mu_1^{(n)}A_{2,1}(\lambda)\big] + h_1^{(r)}\big[\mu_2^{(n)}A_{1,2}(\lambda) + \mu_1^{(n)}A_{2,2}(\lambda)\big],$$

and that the relations (15.14) thereupon show it to reduce to zero at λ_n. At this value of λ the expression (16.5) is, therefore, indeterminate. To fulfill the relation (16.4) it must, however, be continuous. The value to be assigned it at λ_n is, therefore, that which is obtainable by an application of the familiar "l'Hospital's rule," namely, with the use of a superscribing dot to denote a derivative with respect to λ, thus

$$\dot{F}(x, \lambda_n) \equiv \left[\frac{\partial}{\partial\lambda}F(x, \lambda)\right]_{\lambda=\lambda_n}, \tag{16.8}$$

the value

$$\Phi_{n,r}(\lambda_n) = \mu_2^{(n)}\dot{A}_1(U_r, \lambda_n) + \mu_1^{(n)}\dot{A}_2(U_r, \lambda_n). \tag{16.9}$$

For the further analysis of this formula we must distinguish between the case in which the index of λ_n is 1, and that in which it is 2.

If λ_n is of the index 1 we are still concerned with the formula (16.9) only in the case that $r=n$. In this instance the relation

$$A_{\sigma,\tau}(\lambda_n) \neq 0, \tag{16.10}$$

is, moreover, fulfilled for some choice of the subscripts σ, τ, and this implies through the equations (14.9) and (15.14) that

$$h_\tau^{(n)} \neq 0, \qquad \mu_\sigma^{(n)} \neq 0. \tag{16.11}$$

Now the relations (14.4) and (16.1) assure the equation

$$\begin{vmatrix} A_1(U_n, \lambda) & A_2(U_n, \lambda) \\ A_{1,\tau}(\lambda) & A_{2,\tau}(\lambda) \end{vmatrix} = (-1)^\tau h_\tau^{(n)}\Delta(\lambda),$$

and a differentiation of this, together with the evaluations (16.7), shows that

$$\begin{vmatrix} \dot{A}_1(U_n, \lambda_n) & \dot{A}_2(U_n, \lambda_n) \\ A_{1,\tau}(\lambda_n) & A_{2,\tau}(\lambda_n) \end{vmatrix} = (-1)^\tau h_\tau^{(n)}\dot{\Delta}(\lambda_n).$$

The determinant on the left of this equality is, however, that of the system which is comprised of the equation (16.9) with $r=n$, and the equation

$$0 = \mu_2^{(n)}A_{1,\tau}(\lambda_n) + \mu_1^{(n)}A_{2,\tau}(\lambda_n),$$

which is one of the pair (15.14). The eliminant of the coefficient of $A_{\sigma,\tau}(\lambda_n)$ in this system is, therefore, found to be the relation

$$\Phi_{n,n}(\lambda_n) = (-1)^{\sigma+\tau}\frac{h_\tau^{(n)}\mu_\sigma^{(n)}}{A_{\sigma,\tau}(\lambda_n)}\dot{\Delta}(\lambda_n). \tag{16.12}$$

The right-hand member of this is different from zero or equal to zero according as the multiplicity of λ_n as a root of the characteristic equation (14.5) is equal to or greater than 1. It has been shown thus for the case at hand, namely when the index of λ_n is 1, that the relations

$$\Phi_{n,r}(\lambda_r)\begin{cases}= 0, & \text{if } r \neq n,\\ \neq 0, & \text{if } r = n,\end{cases} \tag{16.13}$$

maintain if and only if the multiplicity of λ_n and its index are equal.

Consider now the case of a characteristic value λ_n of the index 2, and let its designations in the array (14.7) be λ_m and λ_{m+1}. The set of equations obtained from the relation (16.9) when n and r are given values from the pair m, $m+1$, yield the determinant relation

$$\begin{vmatrix} \Phi_{m,m}(\lambda_n) & \Phi_{m+1,m}(\lambda_n) \\ \Phi_{m,m+1}(\lambda_n) & \Phi_{m+1,m+1}(\lambda_n)\end{vmatrix} = \begin{vmatrix} \dot{A}_1(U_m, \lambda_n) & \dot{A}_2(U_m, \lambda_n) \\ \dot{A}_1(U_{m+1}, \lambda_n) & \dot{A}_2(U_{m+1}, \lambda_n)\end{vmatrix} \cdot \begin{vmatrix} \overset{(m)}{\mu_2} & \overset{(m+1)}{\mu_2} \\ \overset{(m)}{\mu_1} & \overset{(m+1)}{\mu_1}\end{vmatrix}.$$

Now the formulas (16.6) assure the equation

$$\begin{vmatrix} A_1(U_m, \lambda) & A_2(U_m, \lambda) \\ A_1(U_{m+1}, \lambda) & A_2(U_{m+1}, \lambda)\end{vmatrix} = \begin{vmatrix} \overset{(m)}{h_2} & \overset{(m+1)}{h_2} \\ \overset{(m)}{h_1} & \overset{(m+1)}{h_1}\end{vmatrix} \Delta(\lambda),$$

and a two-fold differentiation of this together with the evaluation (16.7) leads to the equality

$$2\begin{vmatrix} \dot{A}_1(U_m, \lambda_n) & \dot{A}_2(U_m, \lambda_n) \\ \dot{A}_1(U_{m+1}, \lambda_n) & \dot{A}_2(U_{m+1}, \lambda_n)\end{vmatrix} = \begin{vmatrix} \overset{(m)}{h_2} & \overset{(m+1)}{h_2} \\ \overset{(m)}{h_1} & \overset{(m+1)}{h_1}\end{vmatrix} \ddot{\Delta}(\lambda_n).$$

The combination of this result with that above evidently leads to the relation

$$\begin{vmatrix} \Phi_{m,m}(\lambda_n) & \Phi_{m+1,m}(\lambda_n) \\ \Phi_{m,m+1}(\lambda_n) & \Phi_{m+1,m+1}(\lambda_n)\end{vmatrix} = \frac{1}{2}\begin{vmatrix} \overset{(m)}{\mu_2} & \overset{(m+1)}{\mu_2} \\ \overset{(m)}{\mu_1} & \overset{(m+1)}{\mu_1}\end{vmatrix} \cdot \begin{vmatrix} \overset{(m)}{h_2} & \overset{(m+1)}{h_2} \\ \overset{(m)}{h_1} & \overset{(m+1)}{h_1}\end{vmatrix} \ddot{\Delta}(\lambda_n). \tag{16.14}$$

If the multiplicity of λ_n is greater than 2 the right-hand member of the equation (16.14) is zero. It is clear that in that case the equation is contradicted by the relations (16.13) and that these latter, therefore, do not maintain. If, on the other hand, the multiplicity of λ_n is 2, namely the same as its index, no such contradiction is involved. If in this case the parameter pairs μ_1^m, μ_2^m and μ_1^{m+1}, μ_2^{m+1}, which have thus far been left unspecified except that they be linearly independent, are now specified to fulfill the relations

$$\begin{aligned} \overset{(m)}{\mu_2}\dot{A}_1(U_{m+1}, \lambda_n) + \overset{(m)}{\mu_1}\dot{A}_2(U_{m+1}, \lambda_n) &= 0,\\ \overset{(m+1)}{\mu_2}\dot{A}_1(U_m, \lambda_n) + \overset{(m+1)}{\mu_1}\dot{A}_2(U_m, \lambda_n) &= 0,\end{aligned} \tag{16.15}$$

it follows at once through the equations (16.9) that $\Phi_{m,m+1}(\lambda_n) = \Phi_{m+1,m}(\lambda_n) = 0$, and then through the equation (16.14) that $\Phi_{n,n}(\lambda_n)\neq 0$, for $n=m$, $m+1$. The relations (16.13), which by virtue of the equation (16.4) take the form

$$(16.16)\quad \int_a^b q(x)u_r(x)v_n(x)dx + \mu_2^{(n)}B_1(u_r) + \mu_1^{(n)}B_2(u_r)\begin{cases} = 0, & \text{if } r\neq n,\\ \neq 0, & \text{if } r = n,\end{cases}$$

thus maintain whenever the characteristic value λ_n is one whose index and multiplicity are equal, and do not maintain in any other case. These relations are the ones that were sought. They are, namely, those which are expressive of the generalized orthogonality which subsists among the characteristic solutions of two boundary problems that are adjoint.

CHAPTER 17

The formal representation of an arbitrary function. In the instance of any boundary problem which admits of infinitely many characteristic solutions, the physical context in connection with which the problem arises leads in a natural way to the question of the representability of an arbitrary function in terms of these solutions in the manner

$$(17.1)\qquad f(x) = \sum_{r=0}^{\infty} c_r u_r(x).$$

This has already been observed in several instances of the Fourier theory, which is, of course, exemplary of the more general case. As in the trigonometric case, the crux of the formal problem thrown up in this way devolves upon a determination of the coefficients c_r. We shall consider this matter now, not in a rigorous way, but formally. The relation (17.1) will, therefore, be taken to be amenable to all such operations as shall be made upon it, and no consideration will be given to matters of convergence. The deductions will, therefore, of course, be levoid of all power of proof. Their purpose is purely an exploratory one.

To begin with, let the symbols f_i, $i=1, 2$ be used as abbreviations for the expressions $B_i(f)$, as these latter are obtainable from the relations (16.3). The equation (17.1) thus has associated with it the pair of auxiliary relations

$$(17.2)\qquad f_i = \sum_{r=0}^{\infty} c_r B_i(u_r), \qquad i = 1, 2,$$

and in terms of the elements $f(x)$, f_1 and f_2 the formulas

$$(17.3)\qquad I_n(f) \equiv \int_a^b q(x)v_n(x)f(x)dx + \mu_2^{(n)}f_1 + \mu_1^{(n)}f_2, \qquad n = 0, 1, 2, \cdots,$$

may be taken to define their left-hand members. Upon substituting into these formulas the infinite series evaluations (17.1) and (17.2), interchanging the order of the integrations and summations, and collecting the terms in any coefficient c_r, it is found that alternatively

$$I_n(f) = \sum_{r=0}^{\infty} c_r \Phi_{n,r}(\lambda_r).$$

The symbols $\Phi_{n,r}(\lambda_r)$ in this stand, as heretofore, for the left-hand members of the relations (16.16). Since these latter vanish whenever $r \neq n$, the equation reduces to the simple form

$$I_n(f) = c_n \Phi_{n,n}(\lambda_n). \tag{17.4}$$

Let it be assumed now as a hypothesis which is to cover the entire remaining portion of these deductions, that *the boundary problem* (13.10) *in question is one for which there are infinitely many characteristic values, and for which, moreover, each of these is of a multiplicity that is equal to its index.* By the relations (16.13) the equations (17.4) yield, then, for each n, the evaluation

$$c_n = \frac{I_n(f)}{\Phi_{n,n}(\lambda_n)}. \tag{17.5}$$

It will be evident even from the most casual review of the procedure described that it is limited, insofar as actual applicability is concerned, to the functions of a materially restricted class. The expressions $B_i(f)$ are, for instance, significant only for functions that are differentiable at $x = a$ and $x = b$, while other heavy restrictions are manifestly involved. Beyond that it is clear that the result can imply nothing of the representability of a function $f(x)$, since that representability was at the very outset assumed. A theory of representation must, accordingly, be approached differently.

Let it be supposed, therefore, that a function $f(x)$ which is arbitrary except that it possesses certain requisite properties of integrability, and a pair of constants f_1, f_2 are given. The constants are likewise to be regarded as arbitrary. In particular they need have no specific relation to the values of $f(x)$. From these elements $f(x)$, f_1, and f_2 the values $I_n(f)$ are constructible through the formulas (17.3). There is thus associated with them a sequence of constants (17.5), or, in other words, a series of the form (17.1). We shall indicate this association in the manner

$$f(x) \sim \sum_{n=0}^{\infty} \frac{I_n(f) u_n(x)}{\Phi_{n,n}(\lambda_n)}. \tag{17.6}$$

There is no implication at this stage that the series here written down is convergent, or, even should that be the case, that its value is $f(x)$. The continuing theory is to be shaped toward the investigation of those matters. As it stands, the series is to be approached afresh and with no regard for the manner in which it was deduced. Its convergence is to be studied. Any conditions, if such are found, under which the series does converge and to the value $f(x)$, will be conditions under which the symbol of association in the relation (17.6) may be replaced by one of equality and a theory of representation maintains. The direction of our further theoretical developments is thus forecast.

CHAPTER 18

Some examples. Throughout the discussion thus far no references whatsoever to illustrative examples have been made. A stage has now been reached, however, at which the consideration of some explicit cases may be both interesting and instructive. We shall, therefore, interrupt the theoretical developments at this point to review some particular examples with the special purpose of illustrating concretely some of the salient features of the theory as it has thus far been deduced. It will be seen in connection with one of these cases how the present more general theory wholly includes that of Fourier.

Because of the breadth of the basis upon which the theory is built, it is a simple matter to construct in great variety examples that are subsumed under it. In the choice that is set forth below, however, an effort has been made to avoid in as large a measure as possible all such complications as are not germane to the matters essentially at issue. Such could be only distracting and not illuminating. For this reason, in particular, the boundary problems that are drawn upon have all been chosen to be such as involve differential equations that are explicitly solvable. The theory in no way requires that. For convenience in the notation the symbol ρ has been introduced to stand for a square root of λ, thus

$$\rho^2 = \lambda. \tag{18.1}$$

This, however, is to be looked upon merely as an abbreviation. The effective parameter will continue to be λ even when an expression is written in terms of ρ. It will generally be found on this account that a form that is indeterminate as to ρ is quite specific in λ, as is the case with the form $\sin \alpha\rho/\rho$, which near $\lambda = 0$ is to be thought of as defined by the power series

$$\alpha - \frac{\alpha^3}{3!}\lambda + \frac{\alpha^5}{5!}\lambda^2 \cdots .$$

It may also be worth observing in conjunction with the notation defined through the relation (16.8), that

$$\dot{F} = \frac{1}{2\rho}\,\frac{\partial F}{\partial \rho}\,. \tag{18.2}$$

Example 1. The boundary problem

$$\begin{gathered} u'' - \frac{2}{x}u' - \left(\lambda - \frac{2}{x^2}\right)u = 0, \\ u\left(\frac{\pi}{2}\right) = 0, \\ u\left(\frac{3\pi}{2}\right) = 0. \end{gathered} \tag{18.3}$$

The fundamental interval at the ends of which the boundary relations of this problem apply, namely $\pi/2 \leqq x \leqq 3\pi/2$, is one upon which the differential equation has continuous coefficients and no singular points. Beyond that the coefficients $\alpha_{i,j}$ of the boundary relations are the elements of the matrix

$$\begin{Vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix},$$

and this, the matrix (13.8), is of the rank 2 for all values of λ. The boundary problem is thus of the type (13.10). Its differential equation admits as linearly independent solutions analytic in λ, the functions

$$\phi_1(x, \lambda) \equiv x[e^{\rho x} + e^{-\rho x}],$$

$$\phi_2(x, \lambda) \equiv \frac{x}{\rho}[e^{\rho x} - e^{-\rho x}],$$

and as it is formed from these the determinant (14.4) is

$$\Delta(\lambda) \equiv \begin{vmatrix} \frac{\pi}{2}[e^{\rho\pi/2} + e^{-\rho\pi/2}] & \frac{\pi}{2\rho}[e^{\rho\pi/2} - e^{-\rho\pi/2}] \\ \frac{3\pi}{2}[e^{3\rho\pi/2} + e^{-3\rho\pi/2}] & \frac{3\pi}{2\rho}[e^{3\rho\pi/2} - e^{-3\rho\pi/2}] \end{vmatrix}.$$

The evaluation

$$\Delta(\lambda) = \frac{3\pi^2}{2}\left[\frac{e^{\rho\pi} - e^{-\rho\pi}}{\rho}\right],$$

shows that its zeros occur at the points $\rho = \pm ni$, with $n = 1, 2, 3, \cdots$. At each of these points the determinant $\Delta(\lambda)$ is reduced to the rank 1 and $\dot{\Delta}(\lambda)$ is not zero. The characteristic values, which are infinitely many, are thus all of the index and multiplicity 1. They are to be arranged after the fashion (14.7) in the order

$$\lambda_n = -(n+1)^2, \qquad n = 0, 1, 2, \cdots.$$

When r is any even integer, say $r = 2n$, the coefficient of $h_2^{(2n)}$ in the first one of the equations (14.9) is zero. This is seen to determine $h_1^{(2n)}$ to be zero, while leaving $h_2^{(2n)}$ arbitrary (not zero). This latter coefficient may evidently be chosen, therefore, so that the respective characteristic solution, as it is given by the formula (14.10), is explicitly

$$u_{2n}(x) \equiv x \cos (2n+1)x.$$

In a quite similar manner it is found that $h_2^{(2n-1)} = 0$, and that we may accordingly choose

$$u_{2n-1}(x) \equiv x \sin 2nx.$$

By the formulas (15.9) the differential system adjoint to (18.3) is

$$v'' + \frac{2}{x} v' - \lambda v = 0,$$

$$v\left(\frac{\pi}{2}\right) = 0,$$

$$- v'\left(\frac{\pi}{2}\right) - \frac{4}{\pi} v\left(\frac{\pi}{2}\right) = \mu_2,$$

$$- v\left(\frac{3\pi}{2}\right) = 0,$$

$$v'\left(\frac{3\pi}{2}\right) + \frac{4}{3\pi} v\left(\frac{3\pi}{2}\right) = \mu_1.$$

Of the equations in this, the first, the second, and the fourth, effectively define the adjoint boundary problem, while the remaining ones give equivalents of the parameters μ_1 and μ_2. The formulas (15.5) yield as solutions of the differential equation in this system the functions

$$\psi_1(x, \lambda) \equiv \frac{-1}{4\rho x} \left[e^{\rho x} - e^{-\rho x}\right],$$

$$\psi_2(x, \lambda) \equiv \frac{1}{4x} \left[e^{\rho x} + e^{-\rho x}\right].$$

The equations (15.16) show that for any n, $k_1^{2n-1} = 0$ and $k_2^{2n} = 0$. The characteristic solutions $v_n(x)$ may accordingly be taken to be

$$v_{2n-1}(x) \equiv \frac{\sin 2nx}{x},$$

$$v_{2n}(x) \equiv \frac{\cos (2n + 1)x}{x}.$$

The boundary problem (18.3) does not involve λ in its boundary relations. Hence the constants $\beta_{i,j}$ that appear in the formulas (13.7) and (16.3) are all zero, and the forms (16.3) accordingly all vanish. The relation (16.4) thus shows that

$$\Phi_{n,r}(\lambda_r) = - \int_{\pi/2}^{3\pi/2} u_r(x) v_n(x) dx,$$

and the relations of orthogonality (16.16) therefore assume a familiar, purely trigonometrical form. With the special choice of constants $f_1 = 0$, $f_2 = 0$, the formulas (17.3) and (17.6) yield, in association with an arbitrary function $f(x)$, the series of characteristic solutions

$$f(x) \sim c_0 x \cos x + c_1 x \sin 2x + c_2 x \cos 3x + c_3 x \sin 4x + \cdots$$

with the coefficients

$$c_{2n-1} = \frac{2}{\pi}\int_{\pi/2}^{3\pi/2} f(s)\,\frac{\sin 2ns}{s}\,ds,$$

$$c_{2n} = \frac{2}{\pi}\int_{\pi/2}^{3\pi/2} f(s)\,\frac{\cos (2n+1)s}{s}\,ds.$$

Example 2. The boundary problem

$$(18.4)\qquad \begin{aligned} u'' - \frac{1}{x}u' + \frac{16\pi^2x^2}{9}\lambda u &= 0,\\ u(1) &= 0,\\ 2u'(1) - u'(2) &= 0. \end{aligned}$$

This boundary problem is of the form (13.10) on the fundamental interval $1 \leqq x \leqq 2$. Its differential equation admits the solutions

$$\phi_1(x, \lambda) \equiv \cos\left(\frac{2\pi\rho}{3}\,x^2\right),$$

$$\phi_2(x, \lambda) \equiv \frac{1}{\rho}\sin\left(\frac{2\pi\rho}{3}\,x^2\right),$$

and as formed from these functions

$$\Delta(\lambda) \equiv \begin{vmatrix} \cos\left(\frac{2\pi\rho}{3}\right) & \frac{1}{\rho}\sin\left(\frac{2\pi\rho}{3}\right) \\ \frac{8\pi\rho}{3}\left[\sin\left(\frac{8\pi\rho}{3}\right) - \sin\left(\frac{2\pi\rho}{3}\right)\right] & \frac{8\pi}{3}\left[\cos\left(\frac{2\pi\rho}{3}\right) - \cos\left(\frac{8\pi\rho}{3}\right)\right] \end{vmatrix}.$$

The evaluation

$$\Delta(\lambda) = \frac{8\pi}{3}(1 - \cos 2\pi\rho),$$

shows that the zeros occur at the points $\rho = 0\ \pm 1,\ \pm 2, \cdots$. There are thus infinitely many characteristic values, and since at each of them the determinant is reduced to the rank 1 they are all of the index 1. We may accordingly write

$$\lambda_n = n^2, \qquad\qquad n = 0, 1, 2, \cdots.$$

But it is now seen at once that $\dot{\Delta}(\lambda_n) = 0$ for each n. The characteristic values are thus each of a multiplicity that is higher than its index and the boundary problem is therefore excluded by the hypothesis of chapter 17.

Example 3. The boundary problem

$$\begin{aligned} u'' + \lambda u &= 0, \\ u(-\pi) - u(\pi) &= 0, \\ u'(-\pi) - u'(\pi) &= 0. \end{aligned} \tag{18.5}$$

On the interval $-\pi \leqq x \leqq \pi$ this boundary problem is of the form (13.10). We find for it

$$\phi_1(x, \lambda) \equiv \cos \rho x, \qquad \phi_2(x, \lambda) \equiv \frac{\sin \rho x}{\rho}, \tag{18.6}$$

and accordingly

$$\Delta(\lambda) \equiv \begin{vmatrix} 0 & -\dfrac{2}{\rho} \sin \rho\pi \\ 2\rho \sin \rho\pi & 0 \end{vmatrix}. \tag{18.7}$$

Since from this

$$\Delta(\lambda) = 4 \sin^2 \rho\pi, \tag{18.8}$$

its zeros occur for $\rho = 0, \pm 1, \pm 2, \cdots$. At the first of these the rank of the determinant (18.7) is 1, at all others it is 0. The first characteristic value $\lambda_0 = 0$ is thus of the index 1 and the remaining ones are of the index 2. They are accordingly to be arranged thus

$$\lambda_0 = 0, \qquad \lambda_{2n-1} = \lambda_{2n} = n^2, \qquad n = 1, 2, 3, \cdots. \tag{18.9}$$

Each value is found to be of a multiplicity equal to its index. By the relations (14.9) it is shown that $h_1^{(0)} = 0$, so that the first characteristic solution is $u_0(x) \equiv 1$, or any multiple of this. For the other characteristic values the equations (14.9) are vacuous and the coefficients $h_1^{(r)}$ and $h_2^{(r)}$ accordingly arbitrary. We may therefore choose $u_{2n-1}(x)$ and $u_{2n}(x)$ as $\phi_1(x, \lambda_{2n-1})$ and $\phi_2(x, \lambda_{2n})$ respectively,

$$\begin{aligned} u_{2n-1}(x) &\equiv \cos nx, \\ u_{2n}(x) &\equiv \sin nx. \end{aligned} \tag{18.10}$$

The boundary problem (18.5) is readily seen to be self-adjoint. It is, therefore, admissible to choose $v_n(x) \equiv u_n(x)$ for every n. Since the boundary relations are independent of λ the forms $B_i(u_n)$ all vanish, and it is found accordingly that

$$\Phi_{n,r}(\lambda_r) = \int_{-\pi}^{\pi} u_r(x) u_n(x) dx.$$

The relations (16.13) are thus, in this instance, merely expressive of the familiar property of orthogonality of the sines and cosines of multiples of x. With the choice of constants $f_1=0$, $f_2=0$, the formulas (17.3) and (17.5) define the coefficients c_n to be those of Fourier, and the relation (17.6) is simply the association of a function $f(x)$ with its Fourier series.

CHAPTER 19

Another example. Although the examples that have been cited in the chapter above were earmarked among themselves by some conspicuous qualitative dissimilarities, they fail, even when taken together, to illustrate adequately some of the more pronounced departures from classical theory which the generalization permits. The reason for this lies in the fact that the boundary problems that are basic to them all involve only such boundary relations as do not depend upon the parameter λ. In the following a problem of the excepted class is to be briefly analyzed.

Example 4. The boundary problem

$$\begin{aligned} u'' + 4\lambda u &= 0, \\ u(0) &= 0, \\ u'(1) + [\nu^2 + 1 - \lambda]u(1) &= 0, \end{aligned} \tag{19.1}$$

with ν standing for any real constant.

Such boundary problems as this present themselves in connection with a variety of physical investigations of which the following ones may be looked upon as in some measure typical. [18]

Problem (*i*). A right cylindrical solid with a cross-section of any shape and size, and with plane terminal faces at $x=0$ and $x=1$, has its lateral surface insulated against the passage of heat and has an initial distribution of temperatures depending only upon the longitudinal coördinate x. At the time $t=0$ this solid is placed in contact with a quantity of liquid at one of its terminal faces, and the liquid is thereupon kept well stirred to insure that the temperature is uniform throughout it at each instant. The temperature of the liquid and the temperature distribution in the solid at any subsequent time are to be calculated.

Problem (*ii*). A solid metal sphere with an initial distribution of temperatures that is symmetrical about its center, is cooled by being plunged into a mass of liquid. The liquid is kept well stirred. The temperatures of the liquid and those within the solid during the cooling are to be determined.

Problem (*iii*). A mass of material is uniformly distributed at the time $t=0$ throughout a jelly in a cylindrical container. The jelly is covered with a liquid that is kept well stirred. From the experimental measurements of the concentration of material in the liquid as a function of the time, the coefficient of diffusion of the material in the jelly is to be found.

The boundary problem (19.1) is of the form (13.10). Its differential equation admits as solutions the functions

$$\phi_1(x, \lambda) \equiv \cos 2\rho x, \qquad \phi_2(x, \lambda) \equiv \frac{\sin 2\rho x}{\rho}, \tag{19.2}$$

and the evaluation of the determinant

$$\Delta(\lambda) \equiv \begin{vmatrix} 1 & 0 \\ -2\rho \sin 2\rho + (\nu^2+1-\lambda) \cos 2\rho & 2 \cos 2\rho + (\nu^2+1-\lambda) \dfrac{\sin 2\rho}{\rho} \end{vmatrix}, \tag{19.3}$$

gives as the characteristic equation

$$2 \cos 2\rho + (\nu^2 + 1 - \rho^2) \frac{\sin 2\rho}{\rho} = 0. \tag{19.4a}$$

Alternative forms of this are

$$\cot 2\rho = \frac{\rho}{2} - \frac{\nu^2 + 1}{2\rho}, \tag{19.4b}$$

and

$$e^{4\rho i} = \frac{(\rho + i)^2 - \nu^2}{(\rho - i)^2 - \nu^2}. \tag{19.4c}$$

The characteristic values are thus the squares of the roots of a transcendental equation and it is readily seen that they do not admit of expression by any elementary formula. The essential facts concerning them are nevertheless deducible.

Thus if ρ is any complex value with a positive imaginary part, the point $(\rho+i)$ in the complex plane is more distant than the point $(\rho-i)$ from any point on the axis of reals, and hence in particular from the points ν and $-\nu$. Hence

$$|(\rho + i) - \nu| > |(\rho - i) - \nu|,$$
$$|(\rho + i) + \nu| > |(\rho - i) + \nu|,$$

and from these relations it follows that the right-hand member of the equation (19.4c) is greater than 1 in absolute value. But for a value of ρ such as this the left-hand member of that equation has an absolute value that is less than 1. This ρ is, therefore, not one for which the equation is satisfied. A similar argument establishes that same fact for any value of ρ that has a negative imaginary part. The roots of the equation must thus all be real.

Now that being so, it is observable from the equation in the form (19.4b), that its roots are the abscissas of the points common to the graphs

$$y = \cot 2\rho,$$

$$y = \frac{\rho}{2} - \frac{\nu^2 + 1}{2\rho},$$

in the Cartesian (ρ, y) plane. These graphs are easily seen to intersect infinitely often and without being tangent to each other at any intersection point. From this it follows at once that there are infinitely many characteristic values and that they are each of the multiplicity 1. From the general relation (14.6), or from the fact that the determinant (19.3) has a constant non-vanishing element, it is to be seen that every characteristic value is of the index 1.

By the first one of the equations (14.9) it is shown that for each r, $h_2^{(r)} = 0$. The characteristic solutions may, therefore, be taken as the functions

$$(19.5) \qquad u_n(x) \equiv \sin 2\sqrt{\lambda_n}\, x,$$

and since the boundary problem is self-adjoint we may also take $v_n(x) \equiv u_n(x)$. The formulas (16.3) and the boundary relations of the system (15.9) give in this case the evaluations

$$B_1(u_r) = 0, \qquad B_2(u_r) = -u_r(1),$$

$$\mu_2^{(n)} = -v_n'(0), \qquad \mu_1^{(n)} = -v_n(1).$$

It is thus found from the equations (16.4) that

$$\Phi_{n,r}(\lambda_r) = 4\int_0^1 u_r(x)u_n(x)dx + u_r(1)u_n(1).$$

Because of the simple form of the functions $u_n(x)$ the integration in this expression is explicitly possible, and the relations of generalized orthogonality (16.16) are therefore verifiable upon the basis of the equation (19.4a). The formulas (17.3) and (17.5) yield as the coefficients in the association

$$(19.6) \qquad f(x) \sim \sum_{n=0}^{\infty} c_n \sin 2\sqrt{\lambda_n}\, x,$$

the values

$$(19.7) \qquad c_n = \frac{4\int_0^1 f(s)u_n(s)ds - f_1u_n'(0) - f_2u_n(1)}{4\int_0^1 u_n^2(s)ds + u_n^2(1)}.$$

A scrutiny of the formulas (19.7) reveals one of the salient features in which this and the Fourier representations are effectively dissimilar. While the coefficients of the latter are wholly determined by the function $f(x)$, those given by the formulas (19.7) depend also upon the prescribed constants f_1 and f_2. The same function $f(x)$ may thus be associated here with many different representations. This is quite consonant with the nature of the physical problems from which boundary problems of the type (19.1) arise, as may easily be appreciated from a consideration of the problems (i) and (ii) that were cited above. In each of those cases the function $f(x)$ represents the initial temperatures in the solid. The subsequent temperatures therein depend, however, not only upon these but also upon the initial temperature of the fluid into which the solid is immersed or with which it is placed in contact. It is through the constants f_1 and f_2 that this initial fluid temperature comes to account. [19]

CHAPTER 20

The Green's function. A noteworthy feature of the Fourier's series, and one which is almost invariably taken as the point of departure for studies of its convergence properties, is the fact that any initial segment of it may be explicitly summed and hence expressed by a compact formula. This was observed in chapter 12, the summation of the segment $S_N(x)$ as given by the relation (12.7) having been accomplished there by the formula (12.8). It is far from obvious that the advantages inherent in this are retainable in the generalization of the theory, for the derivation of the formula in question is directly and explicitly based upon relationships that are peculiarly trigonometric. We shall show that this may nevertheless be done, the key to the requisite deductions residing in a certain function, the so-called Green's function, which generally plays an important rôle in the theory of boundary problems.

Let λ be taken and retained throughout the deductions of this chapter as *not* a characteristic value of the boundary problem (13.10). This problem then admits of no solution, which is to say that a function which fulfills its boundary relations cannot also satisfy the stipulations of its differential equation for all values of x upon the fundamental interval. Now it is significant that the concession which must be made on the part of the differential equation is very slight, amounting to no more, in fact, than the partial relaxation of its stipulations at only a single point $x=s$ of the interval. The function which fulfills the specifications except at $x=s$ may even be required to be continuous there. The failure will occur in its first derivative, which is subject at this point to an ordinary discontinuity. When this discontinuity is of the proper sign and of the unit magnitude, matters which are adjustable without otherwise affecting the issues, the function is called the Green's function. Inasmuch as it depends upon the location of the point s as well as upon the variables x and λ, it is to be denoted by the symbol $G(x, s, \lambda)$. Its properties, by way of summary, are, then:

(i) As a function of x it satisfies the differential equation of the system (13.10) over each of the intervals $a<x<s$, and $s<x<b$;

(ii) It is continuous in x over the whole interval (a, b), but its derivative is discontinuous at the point $x=s$ to accord with the prescription

$$\frac{\partial}{\partial x} G(x, s, \lambda)\bigg]_{x=s+} - \frac{\partial}{\partial x} G(x, s, \lambda)\bigg]_{x=s-} = 1; \tag{20.1}$$

(iii) It fulfills the boundary relations of the differential system (13.10).

By these properties the function $G(x, s, \lambda)$ is uniquely determined, as the following derivation of its form will incidentally show.

As a solution of the differential equation (13.3) the function is expressible upon each of the intervals (a, s) and (s, b) as a linear combination of the solutions $\phi_j(x, \lambda)$, $j=1, 2$, namely

$$G(x, s, \lambda) = \begin{cases} \gamma_{1,2}\phi_1(x, \lambda) + \gamma_{1,1}\phi_2(x, \lambda), & \text{for} \quad a \leqq x \leqq s, \\ \gamma_{2,2}\phi_1(x, \lambda) + \gamma_{2,1}\phi_2(x, \lambda), & \text{for} \quad s \leqq x \leqq b. \end{cases}$$

The specifications (ii) impose upon these forms the relations

$$\gamma_{2,2}\phi_1(s, \lambda) + \gamma_{2,1}\phi_2(s, \lambda) = \gamma_{1,2}\phi_1(s, \lambda) + \gamma_{1,1}\phi_2(s, \lambda),$$
$$\gamma_{2,2}\phi_1'(s, \lambda) + \gamma_{2,1}\phi_2'(s, \lambda) - \gamma_{1,2}\phi_1'(s, \lambda) - \gamma_{1,1}\phi_2'(s, \lambda) = 1,$$

in other words, a pair of equations which may be solved into the form

$$\gamma_{1,1} = \gamma_{2,1} - \frac{\phi_1(s, \lambda)}{\Omega(s, \lambda)},$$
$$\gamma_{1,2} = \gamma_{2,2} + \frac{\phi_2(s, \lambda)}{\Omega(s, \lambda)}.$$

Because of the formulas (15.5) these relations are alternatively

$$\gamma_{1,1} = \gamma_{2,1} - \psi_2(s, \lambda),$$
$$\gamma_{1,2} = \gamma_{2,2} - \psi_1(s, \lambda).$$

The formulas for $G(x, s, \lambda)$ are, therefore, more explicitly

$$G(x, s, \lambda) = \gamma_{2,2}\phi_1(x, \lambda) + \gamma_{2,1}\phi_2(x, \lambda) + g_1(x, s, \lambda), \tag{20.2}$$

in which the final term is that defined by the relations

$$g_1(x, s, \lambda) = \begin{cases} -\phi_1(x, \lambda)\psi_1(s, \lambda) - \phi_2(x, \lambda)\psi_2(s, \lambda), & \text{for} \quad a \leqq x \leqq s, \\ 0, & \text{for} \quad s \leqq x \leqq b. \end{cases} \tag{20.3}$$

The requirement (iii), that $G(x, s, \lambda)$ fulfill the boundary relations, comes to its expression in terms of the symbols (13.9), (14.3) and (15.11), in the pair of equations

$$\gamma_{2,2}A_{i,1}(\lambda) + \gamma_{2,1}A_{i,2}(\lambda) - \overset{(a)}{A}_{i,1}(\lambda)\psi_1(s, \lambda) - \overset{(a)}{A}_{i,2}(\lambda)\psi_2(s, \lambda) = 0, \qquad i = 1, 2.$$

Their solution determines for the coefficients $\gamma_{2,2}$ and $\gamma_{2,1}$ the evaluations

$$\gamma_{2,j} = \frac{(-1)^{j+1}}{\Delta(\lambda)} \begin{vmatrix} A_{1,j}(\lambda) & \overset{(a)}{A}_{1,1}(\lambda)\psi_1(s, \lambda) + \overset{(a)}{A}_{1,2}(\lambda)\psi_2(s, \lambda) \\ A_{2,j}(\lambda) & \overset{(a)}{A}_{2,1}(\lambda)\psi_1(s, \lambda) + \overset{(a)}{A}_{2,2}(\lambda)\psi_2(s, \lambda) \end{vmatrix}, \qquad j = 1, 2.$$

The substitution of these into the formula (20.2) leads to the result

$$(20.4) \qquad G(x, s, \lambda) = g_1(x, s, \lambda) + \frac{g_2(x, s, \lambda)}{\Delta(\lambda)},$$

in which

$$(20.5) \quad g_2(x, s, \lambda) = -\begin{vmatrix} \phi_1(x, \lambda) & \phi_2(x, \lambda) & 0 \\ A_{1,1}(\lambda) & A_{1,2}(\lambda) & \overset{(a)}{A}_{1,1}(\lambda)\psi_1(s, \lambda) + \overset{(a)}{A}_{1,2}(\lambda)\psi_2(s, \lambda) \\ A_{2,1}(\lambda) & A_{2,2}(\lambda) & \overset{(a)}{A}_{2,1}(\lambda)\psi_1(s, \lambda) + \overset{(a)}{A}_{2,2}(\lambda)\psi_2(s, \lambda) \end{vmatrix}.$$

The Green's function has thus been completely determined.

The method which has in this way been described in connection with the differential system (13.10), may now be applied equally well to the calculation of the Green's function $G_a(x, s, \lambda)$ of the adjoint system (15.9). If this is done it will be found that the two Green's functions are simply related by the equation

$$(20.6) \qquad G_a(x, s, \lambda) \equiv G(s, x, \lambda).$$

The function (20.4) thus operates as the Green's function of the adjoint differential system when its second argument is taken to be the variable and the first one is fixed. The set of its properties enumerated above may thus be extended to include the following ones:

(iv) As a function of s it satisfies the differential equation of the system (15.9) over each of the intervals $a<s<x$ and $x<s<b$;

(v) It is continuous in s over the whole interval (a, b), but its derivative as to s is discontinuous at the point $s=x$ to accord with the prescription

$$(20.7) \qquad \frac{\partial}{\partial s}G(x, s, \lambda)\Big]_{s=x+} - \frac{\partial}{\partial s}G(x, s, \lambda)\Big]_{s=x-} = 1;$$

(vi) As a function of s it fulfills the boundary relations of the differential system (15.9), with appropriate determinations of $\mu_1(x, \lambda)$ and $\mu_2(x, \lambda)$.

It is on the whole simpler to verify these facts than to deduce them. By the formulas (20.4), (20.3) and (20.5), the function $G(x, s, \lambda)$ is evidently expressed as a linear combination of the solutions $\psi_j(s, \lambda)$, $j=1, 2$, on each of the intervals (a, x) and (x, b). The property (iv) is thus assured to it. The forms

$$\phi_1(x, \lambda)\psi_1(s, \lambda) + \phi_2(x, \lambda)\psi_2(s, \lambda),$$
$$\phi_1(x, \lambda)\psi_1'(s, \lambda) + \phi_2(x, \lambda)\psi_2'(s, \lambda),$$

are readily seen from the relations (15.5) and (13.5) to take on the values 0 and 1 respectively at the point $s=x$. By that the properties (v) are implied. Finally the formulas (15.5) and (15.11) may be drawn upon to supply the evaluations

$$A_{i,1}^{(a)}(\lambda)\psi_1(a, \lambda) + A_{i,2}^{(a)}(\lambda)\psi_2(a, \lambda) = \alpha_{i,1}(\lambda), \qquad i = 1, 2.$$

By virtue of them it is clear that

$$G(x, a, \lambda) = \frac{-1}{\Delta(\lambda)} \begin{vmatrix} \phi_1(x, \lambda) & \phi_2(x, \lambda) & 0 \\ A_{1,1}(\lambda) & A_{1,2}(\lambda) & \alpha_{1,1}(\lambda) \\ A_{2,1}(\lambda) & A_{2,2}(\lambda) & \alpha_{2,1}(\lambda) \end{vmatrix}.$$

This, however, is simply the statement that

$$G(x, a, \lambda) = \mu_2(x, \lambda)\alpha_{1,1}(\lambda) + \mu_1(x, \lambda)\alpha_{2,1}(\lambda),$$

namely that the first boundary relation of the system (15.9) is fulfilled, with the parameter values

$$\mu_i(x, \lambda) = \frac{(-1)^i}{\Delta(\lambda)} \begin{vmatrix} \phi_1(x, \lambda) & \phi_2(x, \lambda) \\ A_{i,1}(\lambda) & A_{i,2}(\lambda) \end{vmatrix}, \qquad i = 1, 2. \tag{20.8}$$

By the same procedure it may equally well be shown that the remaining boundary relations are also fulfilled with the same values (20.8). Thus the property (vi) maintains.

Although it will not be relevant to the discussion which follows, a conspicuous property of the Green's function may still be mentioned. This is, namely, that the non-homogeneous differential system

$$L(w, \lambda) = F(x),$$
$$A_1(w, \lambda) = 0,$$
$$A_2(w, \lambda) = 0,$$

which is obviously related to the system (13.10), is solved, whatever the function $F(x)$ may be, by the formula

$$w(x, \lambda) = \int_a^b G(x, s, \lambda)F(s)ds.$$

CHAPTER 21

The residues of the Green's function. If the requirement that λ be fixed is now abandoned and this parameter is, on the contrary, regarded henceforth as a variable that is free to range over the complex plane, it is recognizable from the formulas (20.3) and (20.5) that the functions $g_1(x, s, \lambda)$ and $g_2(x, s, \lambda)$ are analytic in λ. The same is also true of the function $\Delta(\lambda)$. The formula (20.4) therefore shows that the Green's function is analytic except for poles, these latter occurring at and only at the zeros of $\Delta(\lambda)$, namely at the characteristic values. It has been assumed as a part of our general hypothesis that each characteristic value is of a multiplicity equal to its index. As a consequence of that it will be found that for each n the product $(\lambda-\lambda_n)G(x, s, \lambda)$ approaches a finite limit when $\lambda \to \lambda_n$. This limit is known as the *residue* of $G(x, s, \lambda)$ at λ_n. We shall use the prefix "res_n" to designate it, thus

$$\text{res}_n\, G(x, s, \lambda) = \lim_{\lambda \to \lambda_n} (\lambda - \lambda_n) G(x, s, \lambda), \tag{21.1}$$

and shall show that certain residues closely related to these are significantly associated with the terms of the series in a representation (17.6).

Since the function $g_1(x, s, \lambda)$ is analytic in λ when x and s have any specified values, this part of $G(x, s, \lambda)$ evidently makes no contribution to the right-hand member of the relation (21.1). It is, therefore, only the part $g_2(x, s, \lambda)/\Delta(\lambda)$ that needs consideration. Upon expanding the determinant (20.5) by the elements of its last column, therefore, it is found that the relation (21.1) may be more explicitly written as

$$\text{res}_n\, G(x, s, \lambda) = \lim_{\lambda \to \lambda_n} (\lambda - \lambda_n) \sum_{i=1}^{2} E_i(x, \lambda) \left[A_{3-i,1}^{(a)}(\lambda)\psi_1(s, \lambda) + A_{3-i,2}^{(a)}(\lambda)\psi_2(s, \lambda) \right],$$

the functions $E_i(x, \lambda)$ being defined by the formulas

$$E_i(x, \lambda) = \frac{(-1)^i}{\Delta(\lambda)} \begin{vmatrix} \phi_1(x, \lambda) & \phi_2(x, \lambda) \\ A_{i,1}(\lambda) & A_{i,2}(\lambda) \end{vmatrix}, \qquad i = 1, 2. \tag{21.2}$$

From the alternative form

$$\begin{aligned} &\text{res}_n\, G(x, s, \lambda) \\ &\quad = \sum_{i=1}^{2} \left\{ \text{res}_n\, E_i(x, \lambda) \right\} \left[A_{3-i,1}^{(a)}(\lambda_n)\psi_1(s, \lambda_n) + A_{3-i,2}^{(a)}(\lambda_n)\psi_2(s, \lambda_n) \right], \end{aligned} \tag{21.3}$$

it thus follows that the evaluation of the residues of $G(x, s, \lambda)$ may be made to depend upon the determination of residues of the functions (21.2). To make these determinations we must consider separately the case in which the index of λ_n is 1 and that in which it is 2. In doing that it will be convenient to employ the symbol $\delta_{i,j}$ to stand for the "Kronecker delta," namely in the sense

$$\delta_{i,j} = \begin{cases} 1, & \text{if} \quad i = j, \\ 0, & \text{if} \quad i \neq j. \end{cases}$$

Suppose then, to begin with, that λ_n is of the index 1. Then

$$\lim_{\lambda \to \lambda_n} \frac{\Delta(\lambda)}{\lambda - \lambda_n} = \dot{\Delta}(\lambda_n),$$

and there is a pair of subscripts σ, τ, for which the relations (16.10) and (16.11) maintain. It will be clear that we may write

$$(21.4) \qquad (\lambda - \lambda_n)E_i(x, \lambda) = \frac{-(\lambda - \lambda_n)}{\Delta(\lambda)} \begin{vmatrix} \phi_1(x, \lambda) & \phi_2(x, \lambda) & 0 \\ A_{1,1}(\lambda) & A_{1,2}(\lambda) & \delta_{3-i,1} \\ A_{2,1}(\lambda) & A_{2,2}(\lambda) & \delta_{3-i,2} \end{vmatrix}, \quad i = 1, 2,$$

and that therefore

$$(21.5) \qquad \operatorname{res}_n E_i(x, \lambda) = \frac{-1}{\dot{\Delta}(\lambda_n)} \begin{vmatrix} \phi_1(x, \lambda_n) & \phi_2(x, \lambda_n) & 0 \\ A_{1,1}(\lambda_n) & A_{1,2}(\lambda_n) & \delta_{3-i,1} \\ A_{2,1}(\lambda_n) & A_{2,2}(\lambda_n) & \delta_{3-i,2} \end{vmatrix}.$$

This formula may be considerably modified. It will be seen that if it is multiplied on the right by the determinant

$$\begin{vmatrix} h_2^{(n)} & \delta_{1,\tau} & 0 \\ h_1^{(n)} & \delta_{2,\tau} & 0 \\ 0 & 0 & 1 \end{vmatrix},$$

and this operation is then compensated for by dividing out the value of this determinant, $(-1)^\tau h_\tau^{(n)}$, the resulting form is, by virtue of the relations (14.9) and (14.10),

$$\frac{-1}{(-1)^\tau h_\tau^{(n)} \dot{\Delta}(\lambda_n)} \begin{vmatrix} u_n(x) & \phi_\tau(x, \lambda_n) & 0 \\ 0 & A_{1,\tau}(\lambda_n) & \delta_{3-i,1} \\ 0 & A_{2,\tau}(\lambda_n) & \delta_{3-i,2} \end{vmatrix}.$$

The further multiplication of this on the left by the determinant

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & \delta_{1,\sigma} & \delta_{2,\sigma} \\ 0 & \mu_2^{(n)} & \mu_1^{(n)} \end{vmatrix},$$

and the appropriate subsequent division gives it the form

$$\frac{1}{(-1)^{\sigma+\tau} h_{\tau}^{(n)} \mu_{\sigma}^{(n)} \dot{\Delta}(\lambda_n)} \begin{vmatrix} u_n(x) & \phi_\tau(x, \lambda_n) & 0 \\ 0 & A_{\sigma,\tau}(\lambda_n) & \delta_{3-i,\sigma} \\ 0 & 0 & \mu_i^{(n)} \end{vmatrix},$$

by virtue of the relations (15.14). The expansion of this reduces it, because of the equation (16.12), to the evaluation

$$\text{(21.6)} \qquad \operatorname{res}_n E_i(x, \lambda) = \frac{u_n(x)\mu_i^{(n)}}{\Phi_{n,n}(\lambda_n)}, \qquad i = 1, 2,$$

and the substitution of these results into the equation (21.3), together with an application of the relations (15.16) and (15.15), leads to the conclusion that

$$\text{(21.7)} \qquad \operatorname{res}_n G(x, s, \lambda) = \frac{u_n(x)v_n(s)}{\Phi_{n,n}(\lambda_n)}.$$

Finally with the formulas (21.6) and (21.7) at hand it is a simple matter to recognize that the value of

$$\text{(21.8)} \qquad \operatorname{res}_n \left\{ \int_a^b q(s)G(x, s, \lambda)f(s)ds + E_1(x, \lambda)f_2 + E_2(x, \lambda)f_1 \right\},$$

is precisely the term

$$\frac{I_n(f)u_n(x)}{\Phi_{n,n}(\lambda_n)},$$

of the series (17.6).

If λ_n is of the index 2 and appears in the array (14.7) as λ_m and λ_{m+1}, the relation (21.4) is more appropriately taken in the form

$$(\lambda - \lambda_n)E_i(x, \lambda) = \frac{-\begin{vmatrix} \phi_1(x, \lambda) & \phi_2(x, \lambda) & 0 \\ \dfrac{A_{1,1}(\lambda)}{\lambda - \lambda_n} & \dfrac{A_{1,2}(\lambda)}{\lambda - \lambda_n} & \delta_{3-i,1} \\ \dfrac{A_{2,1}(\lambda)}{\lambda - \lambda_n} & \dfrac{A_{2,2}(\lambda)}{\lambda - \lambda_n} & \delta_{3-i,2} \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 1 \\ \dfrac{A_{1,1}(\lambda)}{\lambda - \lambda_n} & \dfrac{A_{1,2}(\lambda)}{\lambda - \lambda_n} & 0 \\ \dfrac{A_{2,1}(\lambda)}{\lambda - \lambda_n} & \dfrac{A_{2,2}(\lambda)}{\lambda - \lambda_n} & 0 \end{vmatrix}}.$$

Since in this instance each element $A_{i,j}(\lambda)$ vanishes at λ_n, it is seen that

$$\lim_{\lambda\to\lambda_n} \frac{A_{i,j}(\lambda)}{\lambda - \lambda_n} = \dot{A}_{i,j}(\lambda_n),$$

and hence that

$$\operatorname{res}_n E_i(x, \lambda) = \frac{-\begin{vmatrix} \phi_1(x, \lambda_n) & \phi_2(x, \lambda_n) & 0 \\ \dot{A}_{1,1}(\lambda_n) & \dot{A}_{1,2}(\lambda_n) & \delta_{3-i,1} \\ \dot{A}_{2,1}(\lambda_n) & \dot{A}_{2,2}(\lambda_n) & \delta_{3-i,2} \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 1 \\ \dot{A}_{1,1}(\lambda_n) & \dot{A}_{1,2}(\lambda_n) & 0 \\ \dot{A}_{2,1}(\lambda_n) & \dot{A}_{2,2}(\lambda_n) & 0 \end{vmatrix}}.$$

Let each one of the determinants in this ratio be multiplied on the right by

$$\begin{vmatrix} h_2^{(m)} & h_2^{(m+1)} & 0 \\ h_1^{(m)} & h_1^{(m+1)} & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

Because of the relations (14.10) and (16.6) it then takes the form

$$\frac{-\begin{vmatrix} u_m(x) & u_{m+1}(x) & 0 \\ \dot{A}_1(U_m, \lambda_n) & \dot{A}_1(U_{m+1}, \lambda_n) & \delta_{3-i,1} \\ \dot{A}_2(U_m, \lambda_n) & \dot{A}_2(U_{m+1}, \lambda_n) & \delta_{3-i,2} \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 1 \\ \dot{A}_1(U_m, \lambda_n) & \dot{A}_1(U_{m+1}, \lambda_n) & 0 \\ \dot{A}_2(U_m, \lambda_n) & \dot{A}_2(U_{m+1}, \lambda_n) & 0 \end{vmatrix}},$$

and if this is again modified by multiplying each of the determinants on the left by the factor

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & \mu_2^{(m+1)} & \mu_1^{(m+1)} \\ 0 & \mu_2^{(m)} & \mu_1^{(m)} \end{vmatrix},$$

the result, as a consequence of the relations (16.9) and (16.13), is

$$\frac{-\begin{vmatrix} u_m(x) & u_{m+1}(x) & 0 \\ 0 & \Phi_{m+1,m+1}(\lambda_n) & \mu_i^{(m+1)} \\ \Phi_{m,m}(\lambda_n) & 0 & \mu_i^{(m)} \end{vmatrix}}{\begin{vmatrix} 0 & 0 & 1 \\ 0 & \Phi_{m+1,m+1}(\lambda_n) & 0 \\ \Phi_{m,m}(\lambda_n) & 0 & 0 \end{vmatrix}}.$$

Upon expansion this yields the formula

$$(21.9)\qquad \operatorname{res}_n E_i(x, \lambda) = \frac{u_m(x)\mu_i^{(m)}}{\Phi_{m,m}(\lambda_n)} + \frac{u_{m+1}(x)\mu_i^{(m+1)}}{\Phi_{m+1,m+1}(\lambda_n)}, \qquad i = 1, 2,$$

in accordance with which the relation (21.4) assumes the explicit form

$$(21.10)\qquad \operatorname{res}_n G(x, s, \lambda) = \frac{u_m(x)v_m(s)}{\Phi_{m,m}(\lambda_m)} + \frac{u_{m+1}(x)v_{m+1}(s)}{\Phi_{m+1,m+1}(\lambda_{m+1})}.$$

Thus when λ_n is of the index 2 the residue (21.8) is the sum

$$\frac{I_m(f)u_m(x)}{\Phi_{m,m}(\lambda_m)} + \frac{I_{m+1}(f)u_{m+1}(x)}{\Phi_{m+1,m+1}(\lambda_{m+1})},$$

of the pair of terms which appear in the series (17.6) in association with λ_n.

From these conclusions it may now be seen that if the points representing the characteristic values are plotted in the complex plane, and if the first $(N+1)$ of them taken in the order of increasing distance from the origin are enclosed by a curve C_N, the segment $S_N(x)$ of the series (17.6) that is made up of terms that are associated with these characteristic values is given by the formula

$$S_N(x) = \sum_{n=0}^{N} \operatorname{res}_n \left\{ \int_a^b q(s)G(x, s, \lambda)f(s)ds + E_1(x, \lambda)f_2 + E_2(x, \lambda)f_1 \right\}.$$

Such a sum of residues is, however, familiarly expressible as a contour integral with respect to λ over the curve C_N, namely as

$$(21.11)\quad S_N(x) = \frac{1}{2\pi i}\int_{C_N} \left\{ \int_a^b q(s)G(x, s, \lambda)f(s)ds + E_1(x, \lambda)f_2 + E_2(x, \lambda)f_1 \right\} d\lambda.$$

This is the formula in the general theory that stands in the place of the Fourier formula (12.8)

CHAPTER 22

The Fourier's series again. The formula (21.11) is useful, like its specialized counterpart (12.8), as the natural medium through which an investigation of the associated representations of arbitrary functions may be made. As the contour of integration C_N is successively taken to include more and more characteristic values the formula sums more and more terms of the series, and the approach of the integral to a limit reflects the convergence of the series. An identification of the function (if any) which the series represents thus becomes accessible through a study of the formula's integrand, more particularly through an analysis of the Green's function and the functions (21.2) when $|\lambda|$ is large.

The exposition of the complete analysis that would be requisite for the general case would at this point go well beyond the bounds which have been set for the scope of this discussion. The character of the functions $\phi_1(x, \lambda)$, $\phi_2(x, \lambda)$ as these depend upon λ would need to be determined. In general, however, no explicit formulas for these functions are available, since the differential equation (13.3) is not ordinarily solvable. The difficulties which this circumstance interposes will be manifest, although, to be sure, they are not insurmountable. So-called asymptotic forms for the solutions of an equation of the type (13.3) are deducible by well established methods, and these are entirely adequate to the requirements. This theory of asymptotic solutions, however, we do not propose to go into here. Without it, a restriction of the discussion to considerably narrower confines than have hitherto been observed is in the nature of things inevitable. We shall yield to the necessity by limiting the exposition henceforward to the basis of an outright, albeit a wholly typical, specialization. In fact, therefore, the further investigation is to take the form of an extended analysis of the example 3 of chapter 18. It will be recalled that the theory of the boundary problem basic to that example is none other than the theory of Fourier's series.

For the boundary problem (18.5), and with the choice of constants $f_1=0$, $f_2=0$, the special form of the formula (21.11) to which the attention is to be given is

$$S_N(x) = \frac{1}{2\pi i}\int_{C_N}\int_{-\pi}^{\pi} G(x, s, \lambda)f(s)\,ds\,d\lambda. \tag{22.1}$$

The functions $\phi_1(x, \lambda)$, $\phi_2(x, \lambda)$ may be chosen as those that are given by the relations (18.6), and these lead through the formulas (15.11) and (15.5) to the evaluations

$$A_{1,1}^{(a)}(\lambda) = \cos\rho\pi, \qquad A_{1,2}^{(a)}(\lambda) = -\frac{\sin\rho\pi}{\rho},$$
$$A_{2,1}^{(a)}(\lambda) = \rho\sin\rho\pi, \qquad A_{2,2}^{(a)}(\lambda) = \cos\rho\pi,$$

and

$$\psi_1(s, \lambda) \equiv -\frac{1}{\rho}\sin\rho s, \qquad \psi_2(s, \lambda) \equiv \cos\rho s.$$

The formula (20.5) is thus explicitly

$$g_2(x, s, \lambda) = -\begin{vmatrix} \cos\rho x & \frac{1}{\rho}\sin\rho x & 0 \\ 0 & -\frac{2}{\rho}\sin\rho\pi & -\frac{1}{\rho}\sin\rho(\pi+s) \\ 2\rho\sin\rho\pi & 0 & \cos\rho(\pi+s) \end{vmatrix},$$

and this with the relations (20.3) and (20.4) yields the form

$$(22.2)\qquad G(x, s, \lambda) = \frac{\cos \rho(\pi + s - x)}{2\rho \sin \rho\pi} - \begin{cases} \dfrac{\sin \rho(x - s)}{\rho}, & \text{when } -\pi \leqq x \leqq s, \\ 0, & \text{when } \quad s \leqq x \leqq \pi. \end{cases}$$

In the complex λ-plane the characteristic values are located at the points n^2, with $n=0, 1, 2, 3, \cdots$. The contour C_N, which must enclose precisely the first $(N+1)$ of these, may, therefore, be chosen as the circle centered at the origin and of the radius $(N+\frac{1}{2})^2$. In the ρ-plane the semi-circle Γ_N which is centered at $\rho=0$, which has the radius $(N+\frac{1}{2})$, and upon which $0 \leqq \arg \rho < \pi$, is an image of the circle C_N under the mapping defined by the relation (18.1). The formula (22.1) with its integration expressed in terms of ρ is, therefore,

$$(22.3)\qquad S_N(x) = \frac{1}{\pi i}\int_{\Gamma_N}\int_{-\pi}^{\pi} \rho G(x, s, \lambda) f(s)\, ds\, d\rho.$$

When the point x at which the sum $S_N(x)$ is to be considered lies in the interior of the interval $(-\pi, \pi)$, it is advantageous to take the formula (22.2) in the equivalent form

$$(22.4)\qquad G(x, s, \lambda) = \frac{-i}{2\rho} e^{|s-x|\rho i} + \frac{e^{\pi\rho i}\cos \rho(s - x)}{2\rho \sin \rho\pi},$$

the symbol $|s-x|$ standing as usual for $(s-x)$ or $(x-s)$ according as $x \leqq s$ or $x > s$. The formula (22.3) then assumes the form

$$(22.5)\qquad S_N(x) = \frac{-1}{2\pi}\int_{\Gamma_N}\int_{-\pi}^{\pi} e^{|s-x|\rho i} f(s)\, ds\, d\rho + \int_{-\pi}^{\pi} \Psi(s, x, N) f(s)\, ds,$$

with the function $\Psi(s, x, N)$ defined by the formula

$$(22.6)\qquad \Psi(s, x, N) = \frac{1}{2\pi i}\int_{\Gamma_N} \frac{e^{\pi\rho i}\cos \rho(s - x)}{\sin \rho\pi}\, d\rho.$$

Since this latter function is explicitly integrable as to s, it is found at once that for any choice of (α, β) as a sub-interval of the range $(-\pi, \pi)$

$$(22.7)\qquad \begin{aligned} \int_{\alpha}^{\beta} \Psi(s, x, N)\, ds = {} & \frac{1}{2\pi i}\int_{\Gamma_N} \frac{e^{\pi\rho i}\sin \rho(\beta - x)}{\rho \sin \rho\pi}\, d\rho \\ & - \frac{1}{2\pi i}\int_{\Gamma_N} \frac{e^{\pi\rho i}\sin \rho(\alpha - x)}{\rho \sin \rho\pi}\, d\rho. \end{aligned}$$

The integrals which appear in this formula closely resemble that in the formula (22.6) since the sine and cosine functions are effectively similar in structure. The analysis of the one, which is to be set forth, will therefore be found obviously and readily adaptable to the others also. In the event that the point x is an end point of the interval $(-\pi, \pi)$ a somewhat different grouping of terms in the formulas (22.4) and (22.5) is advantageous. The analysis is, however, essentially similar to that which applies when $-\pi<x<\pi$, and that being so we shall content ourselves with the discussion of this latter case alone.

When it is expressed entirely in terms of exponentials the formula (22.6) may be made to appear in the form

$$\Psi(s, x, N) = \frac{-1}{2}\int_{\Gamma_N} \frac{e^{[2\pi+(s-x)]\rho i} + e^{[2\pi-(s-x)]\rho i}}{\pi(1 - e^{2\pi\rho i})}\, d\rho. \tag{22.8}$$

Even relatively crude appraisals of the functions which appear in the integrand of this yield some significant results. Under the resolution $\rho=\xi+i\eta$, with ξ and η real, the equation of the arc Γ_N is

$$\xi^2 + \eta^2 = (N + \tfrac{1}{2})^2, \qquad \eta \geqq 0,$$

and from this it is readily seen that each one of its points is either one for which

$$N + \tfrac{1}{4} \leqq |\xi| \leqq N + \tfrac{1}{2},$$

or else one for which

$$\eta > \tfrac{1}{4}\sqrt{3}.$$

At every point of the first one of these categories $\cos 2\pi\xi<0$, and the value $|1-e^{2\pi\rho i}|$ which is explicitly

$$\left\{1 - e^{-2\pi\eta}\cos 2\pi\xi + e^{-4\pi\eta}\right\}^{1/2}, \tag{22.9}$$

thus clearly exceeds 1. At any point of the second category the value (22.9) exceeds $(1-e^{-\pi\sqrt{3}/4})$, and is thus *ipso facto* greater than $1/\pi$. The relation

$$\pi\left|1 - e^{2\pi\rho i}\right| > 1, \tag{22.10}$$

thus maintains over the whole arc Γ_N.

Consider now the values

$$\left|e^{[2\pi\pm(s-x)]\rho i}\right|, \tag{22.11}$$

with s and x both upon the interval $(-\pi, \pi)$ and ρ still upon Γ_N. For all such s and x the relation

$$2\pi \pm (s - x) \geqq \pi - |x|,$$

is easily verified, and since the formula $\rho=(N+\frac{1}{2})e^{i\theta}$, in which θ stands for arg ρ, shows that the real part of ρi is $-(N+\frac{1}{2})\sin\theta$, it is seen that the values (22.11) are both less than

$$e^{-[\pi-|x|](N+\frac{1}{2})\sin\theta}$$

Since this latter value is increased by the substitution of $\frac{1}{2}\theta$ for $\sin\theta$ when $0 \leqq \theta \leqq \pi/2$ and of $\frac{1}{2}(\pi-\theta)$, for $\sin\theta$ when $\pi/2 \leqq \theta \leqq \pi$, we may evidently draw from the formula (22.8) the relation

$$|\Psi(s, x, N)| < \int_0^{\pi/2} e^{-[\pi-|x|](N+\frac{1}{2})\frac{1}{2}\theta}(N+\tfrac{1}{2})d\theta + \int_{\pi/2}^{\pi} e^{-[\pi-|x|](N+\frac{1}{2})\frac{1}{2}(\pi-\theta)}(N+\tfrac{1}{2})d\theta.$$

By explicit integrations this reduces to the inequality

$$(22.12) \qquad |\Psi(s, x, N)| < \frac{4}{\pi - |x|}.$$

The companion result

$$(22.13) \qquad \left|\int_\alpha^\beta \Psi(s, x, N)ds\right| < \frac{8}{(\pi - |x|)(N+\frac{1}{2})},$$

may be drawn by similar reasoning from the formula (22.7).

These conclusions are significant. In accordance with them the function $\Psi(s, x, N)$ is uniformly bounded as to s and N, and is such that its integral as to s converges to zero with $1/N$ uniformly as to the interval of integration. These properties, however, are precisely those which were invoked in chapter 12 as being sufficient to insure the relation (12.10) for an arbitrary integrable function $F(s)$. In the present instance, therefore, we may similarly conclude upon the basis of them that the final integral in the equation (22.5) converges to zero. Since the remaining integral in that equation may be evaluated thus

$$-\frac{1}{2\pi}\int_{\Gamma_N}\int_{-\pi}^{\pi} e^{|s-x|\rho i}f(s)dsd\rho = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\,\frac{\sin(N+\frac{1}{2})(s-x)}{s-x}\,ds,$$

it is clear that the equation (22.5) implies for every integrable function $f(x)$ the relation

$$\lim_{N\to\infty}\left[S_N(x) - \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\,\frac{\sin(N+\frac{1}{2})(s-x)}{s-x}\,ds\right] = 0.$$

This is the relation (12.11) already familiar, and through it the reference of the sum $S_N(x)$ to the Dirichlet integral has evidently again been accomplished. The method of its accomplishment here, however, by contrast with that of chapter 12, is one of very general applicability. In particular it is one which in no way depends upon the special trigonometric combination formulas. With adaptations which essentially concern only details, this method is adequate to the analysis of the general series (17.6), namely to the representations of arbitrary functions in terms of the characteristic solutions of any properly constructed boundary problem [20].

References
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REFERENCES

1. Van Vleck, E. B., The influence of Fourier's Series upon the development of Mathematics. Science, vol. 39, 1914, pp. 113–124.

2. Burkhardt, H., Entwicklungen nach oscillierenden Functionen und Integration der Differentialgleichungen der mathematischen Physik. Jahresbericht der Deutschen Mathematiker=Vereinigung. Leipzig. 1908, p. 25 and p. 3.

3. Burkhardt, H., *loc. cit.*, pp. 1–5.

4. Burkhardt, H., *loc. cit.*, p. 8.

5. Burkhardt, H., *loc. cit.*, pp. 10–13.

Cantor, M., Vorlesungen über Geschichte der Mathematik, vol. 3 (2nd ed.), Leipzig, p. 384 and pp. 900–902.

6. Riemann, B., Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe. Gesammelte Werke, Leipzig, 1892, pp. 227–231.

Burkhardt, H., *loc. cit.*, pp. 14–24.

Van Vleck, E. B., *loc. cit.*, pp. 114, 115.

Gibson, G. A. On the History of the Fourier Series. Proceedings of the Edinburgh Mathematical Society, vol. XI, 1893, pp. 139–142.

7. Burkhardt, H., *loc. cit.*, pp. 29–31.

8. Burkhardt, H., *loc. cit.*, pp. 27–43.

Riemann, B., *loc. cit.*, pp. 233–234.

9. Burkhardt, H., *loc. cit.*, pp. 70–71.

10. Fourier, Joseph, The Analytical Theory of Heat (trans. by A. Freeman), Cambridge, 1878, pp. 104–115.

11. Fourier, *loc. cit.*, p. 168.

12. Fourier, *loc cit.*, p. 184.

13. Ford, L. R., Differential Equations, New York, 1933, p. 75.

14. Hobson, E. W., The Theory of Functions of a Real Variable, vol. II, 2nd ed. Cambridge, 1926, pp. 422, 423.

15. Hobson, *loc. cit.*, p. 482. Knopp, K. Theorie und Anwendungen der Unendlichen Reihen, Berlin, 1922, p. 337.

16. Ford, *loc. cit.*, pp. 66–68.

17. Ince, E. L., Ordinary Differential Equations, London, 1927, p. 73.

18. Langer, R. E. A Problem in Diffusion or in the Flow of Heat for a Solid in Contact with a Fluid. The Tôhoku Mathematical Journal, vol. 35, 1932, pp. 260–275.

19. A more detailed discussion of these problems, and references to the literature are to be found in the paper cited as reference [18].

20. The reference [18] includes a proof of the convergence of the representations of functions in terms of the solutions of the boundary problem (19.1). Both interior and end points of the interval are there considered.

Appendices
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APPENDICES

I. The solution of the system of equations

$$\begin{aligned} u_0 &= 0, \\ u_{k+1} + qu_k + u_{k-1} &= 0, \qquad k = 1, 2, \cdots, (n-1), \\ u_n &= 0. \end{aligned} \tag{I.1}$$

If, in terms of unspecified constants α and β, a solution of the system is sought in the form

$$u_k = \alpha^k - \beta^k, \tag{I.2}$$

it is found upon substitution into the equations to be requisite that

$$\alpha^{k-1}[\alpha^2 + q\alpha + 1] - \beta^{k-1}[\beta^2 + q\beta + 1] = 0, \qquad k = 1, 2, \cdots, (n-1). \tag{I.3}$$

These conditions are fulfilled if α and β are roots of the equation

$$x^2 + qx + 1 = 0,$$

namely if

$$\alpha + \beta = -q, \qquad \alpha\beta = 1. \tag{I.4}$$

Since β must thus be the reciprocal of α, whereas the form (I.2) must vanish when $k=n$, it is seen to be necessary that

$$\alpha^n - \alpha^{-n} = 0,$$

namely that

$$\alpha = e^{\nu\pi i/n}, \qquad \beta = e^{-\nu\pi i/n},$$

with an integral value of ν. For such an index ν the relations (I.4) and (I.2) show that q and u_k have the values

$$q_\nu = -\left(e^{\nu\pi i/n} + e^{-\nu\pi i/n}\right) = -2\cos\frac{\nu\pi}{n}, \tag{I.5}$$

and

$$u_{\nu,k} = \left(e^{k\nu\pi i/n} - e^{-k\nu\pi i/n}\right) = 2i\sin\frac{k\nu\pi}{n},$$

respectively. Since the system under consideration is homogeneous, any multiple of a solution is also such. Hence we may write

$$u_{\nu,k} = A_\nu \sin\frac{k\nu\pi}{n}, \qquad k = 0, 1, 2, \cdots, n, \tag{I.6}$$

with the coefficient A_ν arbitrary.

For the indices $\nu=1, 2, \cdots, (n-1)$, the characteristic values of q, as given by the formula (I.5), are distinct. There can be no others, for by inspection the determinant of the system of equations is a polynomial of the degree $(n-1)$ in q.

II. A proof of the relation

$$\sin \nu\xi = \sin \xi \cdot p_{\nu-1}(\cos \xi), \tag{II.1}$$

with $p_{\nu-1}$ a polynomial of the degree $(\nu-1)$.

If in Demoivre's formula

$$(\cos \nu\xi + i \sin \nu\xi) = (\cos \xi + i \sin \xi)^{\nu},$$

the right-hand member is expanded by the binomial theorem, each resulting term that involves i to an odd power also involves $\sin \xi$ to such a power. Upon equating the pure imaginary components on the two sides of the equation it is thus found that

$$\sin \nu\xi = \sin \xi \cdot Q(\cos \xi, \sin^2 \xi),$$

each term of Q being of the degree $(\nu-1)$ in $\cos \xi$ and $\sin \xi$, and of even degree in $\sin \xi$. By the substitution of $(1-\cos^2\xi)$ for $\sin^2\xi$ the form (II.1) results.

III. A deduction of the identity

$$\cos^j x = \frac{1}{2^{j-1}} \sum_{\mu=0}^{[j/2]} \binom{j}{\mu} \cos (j - 2\mu)x. \tag{III.1}$$

If in the familiar equality

$$\cos x = \tfrac{1}{2}(e^{ix} + e^{-ix}), \tag{III.2}$$

each member is raised to the jth power and the one on the right is then expanded by the binomial theorem, the result is the relation

$$\cos^j x = \frac{1}{2^j} \sum_{\mu=0}^{j} \binom{j}{\mu} e^{(j-2\mu)ix}, \tag{III.3}$$

in which the symbol $\binom{j}{\mu}$ designates the coefficient of a^{μ} in the expansion of $(1+a)^j$. Since $\binom{j}{j-\mu}=\binom{j}{\mu}$, the formula (III.3) may be written alternatively as

$$\cos^j x = \frac{1}{2^j} \sum_{\mu=0}^{[j/2]} \binom{j}{\mu} \left\{ e^{(j-2\mu)ix} + e^{-(j-2\mu)ix} \right\},$$

with $[j/2]$ denoting the largest integer not exceeding $j/2$. By invoking the relation (III.2) again this may be given the form (III.1).

IV. A derivation of the evaluations

$$\text{(IV.1)} \qquad \sum_{\mu=1}^{n-1} \cos\frac{\mu s\pi}{n} = \begin{cases} n - \frac{1}{2}(1+\cos s\pi), & \text{if} \quad s \equiv 0 \pmod{2n}, \\ \quad - \frac{1}{2}(1+\cos s\pi), & \text{if} \quad s \not\equiv 0 \pmod{2n}. \end{cases}$$

The first one of these evaluations is obvious. For when $s \equiv 0 \pmod{2n}$ each term in the sum on the left has the value 1 while the right-hand member is $(n-1)$. If $s \not\equiv 0$ (mod. $2n$) the relation

$$\cos\frac{\mu s\pi}{n} = \frac{1}{2}\left(e^{\mu s\pi i/n} + e^{-\mu s\pi i/n}\right)$$

leads at once to the equation

$$\sum_{\mu=1}^{n-1} \cos\frac{\mu s\pi}{n} = \frac{1}{2} \sum_{\mu=-n+1,\mu\neq 0}^{n-1} e^{\mu s\pi i/n},$$

or, as it may equally well be written, to

$$\text{(IV.2)} \qquad \sum_{\mu=1}^{n-1} \cos\frac{\mu s\pi}{n} = \frac{1}{2}\left[-1 + \sum_{\mu=-n+1}^{n-1} \left(e^{s\pi i/n}\right)^{\mu}\right].$$

The sum in the right-hand member of this is a geometric progression. It is summed, therefore, by the formula

$$\sum_{\mu=-n+1}^{n-1} \left(e^{s\pi i/n}\right)^{\mu} = \frac{\left(e^{s\pi i/n}\right)^{-n+1} - \left(e^{s\pi i/n}\right)^{n}}{1 - \left(e^{s\pi i/n}\right)},$$

and since

$$\left(e^{s\pi i/n}\right)^{\pm n} = (-1)^s = \cos s\pi,$$

its value reduces to $-\cos s\pi$. The formula (IV.2) thus takes on the form of the second evaluation (IV.1).

V. An evaluation of the determinant

$$\text{(V.1)} \qquad D(x) \equiv \begin{vmatrix} 1 & 2 & \cdots (\nu-1) & x \\ 1^3 & 2^3 & \cdots (\nu-1)^3 & x^3 \\ 1^5 & 2^5 & \cdots (\nu-1)^5 & x^5 \\ \cdot & \cdot & \cdots & \cdot \\ 1^{2\nu-1} & 2^{2\nu-1} & \cdots (\nu-1)^{2\nu-1} & x^{2\nu-1} \end{vmatrix}.$$

If in this determinant the elements of each row, beginning with the last one and extending in turn back to the second one, are modified by subtracting from them x^2 times the corresponding elements of the preceding row, the value of the determinant is unchanged. In its new form, however, all but the first element in the last column are zeros, and hence, upon an expansion by the elements of this column, it is found that

$$D(x) = (-1)^{\nu-1}x \begin{vmatrix} 1(1^2-x^2) & 2(2^2-x^2) & \cdots & [\nu-1]([\nu-1]^2-x^2) \\ 1^3(1^2-x^2) & 2^3(2^2-x^2) & \cdots & [\nu-1]^3([\nu-1]^2-x^2) \\ \cdots & \cdots & \cdots & \cdots \\ 1^{2\nu-3}(1^2-x^2) & 2^{2\nu-3}(2^2-x^2) & \cdots & [\nu-1]^{2\nu-3}([\nu-1]^2-x^2) \end{vmatrix}.$$

As it now stands, the factor (n^2-x^2) is common to the elements of the nth column for $n=1, 2, \cdots, (\nu-1)$. Upon factoring these from the determinant the evaluation

$$\text{(V.2)} \qquad D(x) = (-1)^{\nu-1}x\left[\prod_{n=1}^{\nu-1}(n^2-x^2)\right]D_{\nu,\nu},$$

is obtained, $D_{\nu,\nu}$ designating the cofactor of the element in the νth row and νth column of the original form of $D(x)$.

VI. A formal deduction of the relation

$$\text{(VI.1)} \qquad \prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right) = \frac{\sin \pi x}{\pi x}.$$

The roots of the equation $z^{2k}=1$ are obviously $z=\pm 1$ and $z=e^{\pm n\pi i/k}$, for $n=1, 2, \cdots, (k-1)$. The factorization

$$z^{2k}-1 = (z^2-1)\prod_{n=1}^{k-1}\left\{(z-e^{n\pi i/k})(z-e^{-n\pi i/k})\right\},$$

is, therefore, proper. It is, however, clearly equivalent to

$$z^{2k}-1 = (z^2-1)\prod_{n=1}^{k-1}\left(z^2+1-2z\cos\frac{n\pi}{k}\right),$$

a relation from which it follows that

$$\frac{z^k-z^{-k}}{2i} = \left(\frac{z-z^{-1}}{2i}\right)\prod_{n=1}^{k-1}\left(z+z^{-1}-2\cos\frac{n\pi}{k}\right).$$

Now if in this the quantity $e^{i\pi x/k}$ is substituted for z, the equation becomes

$$\sin \pi x = \sin\frac{\pi x}{k}\prod_{n=1}^{k-1}\left[2\cos\frac{\pi x}{k} - 2\cos\frac{\pi n}{k}\right],$$

or, since

$$\cos\frac{\pi x}{k} - \cos\frac{\pi n}{k} = 2\left(\sin^2\frac{\pi n}{2k} - \sin^2\frac{\pi x}{2k}\right),$$

$$\text{(VI.2)} \quad \frac{\sin \pi x}{\pi x} = \frac{\sin\left(\dfrac{\pi x}{k}\right)}{\pi x} \left\{2^{2k-2} \prod_{n=1}^{k-1} \sin^2 \frac{\pi n}{2k}\right\} \prod_{n=1}^{k-1} \left\{1 - \left\{\frac{\sin \dfrac{\pi x}{2k}}{\sin \dfrac{\pi n}{2k}}\right\}^2\right\}.$$

This is an identity. Its limiting form as $x \to 0$ must, therefore, maintain, namely

$$1 = \frac{1}{k}\left\{2^{2k-2} \prod_{n=1}^{k-1} \sin^2 \frac{\pi n}{2k}\right\},$$

an evaluation by virtue of which the relation (VI.2) may itself be reduced to the form

$$\frac{\sin \pi x}{\pi x} = \frac{\sin\left(\dfrac{\pi x}{k}\right)}{\left(\dfrac{\pi x}{k}\right)} \prod_{n=1}^{k-1} \left\{1 - \left\{\frac{\sin \dfrac{\pi x}{2k}}{\sin \dfrac{\pi n}{2k}}\right\}^2\right\}.$$

This is valid for all k, and we may, therefore, permit this index to become infinite. The evaluation (VI.1) formally results.

VII. Establishment of the formula

$$\text{(VII.1)} \quad 1 + \sum_{\nu=1}^{n} 2 \cos \nu\theta = \frac{\sin\left[\dfrac{2n+1}{2}\theta\right]}{\sin\left[\frac{1}{2}\theta\right]}, \quad \text{for} \quad \theta \not\equiv 0 \pmod{2\pi}.$$

Upon substitution of the relations

$$2 \cos \nu\theta = e^{\nu\theta i} + e^{-\nu\theta i},$$

into the left-hand member of the formula (VII.1), this latter is found to be expressible as

$$\sum_{\nu=-n}^{n} e^{\nu\theta i}.$$

This is a geometric progression whose sum, if $\theta \not\equiv 0 \pmod{2\pi}$ is

$$\frac{e^{-n\theta i} - e^{(n+1)\theta i}}{1 - e^{\theta i}}.$$

If in this the numerator and the denominator are each divided by the factor $-2ie^{\theta i/2}$, the fraction assumes the form of the right-hand member of the formula (VII.1). This latter is, therefore, established.

Back Matter

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