Purdue University MA 520
Fourier Analysis and Boundary Value Problems
Spring 2003, Test One

(Instructor: Aaron N. K. Yip)

Name: ____________________________ (Department/Company: ____________)

- This test booklet has FOUR QUESTIONS, totaling 40 points for the whole test. You have 50 minutes to do this test. **Plan your time well. Read the questions carefully. You do not need to attempt the questions in sequence.**

- This test is **open note** but **closed book**. No photocopy of any book pages or chapter. All the note should be prepared by yourself. **No calculator** is allowed.

- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.

- You can use both sides of the papers to write your answers. But please indicate so if you do.

1. (a) Let $u(x,t)$ solve the following diffusion-transport equation:

$$u_t + cu_x = u_{xx}$$

Let $v(x,t) = u(x+ct,t)$. Prove that $v(x,t)$ satisfies the linear heat equation:

$$v_t = v_{xx}$$

(b) Let $u(x,t)$ solve that following Burgers equation:

$$u_t = u_{xx} + (u_x)^2$$

Let the function v be defined by $u = \ln v$, i.e. $v = e^u$. Show that v satisfies a linear heat equation:

$$v_t = v_{xx}$$

(Note: The above two parts are *unrelated* to each other.)

(a) $v(x,t) = u(x+ct, t)$

$$\left.\begin{aligned} v_t &= u_t^{(x+ct,t)} + c\, u_x(x+ct,t) \\ v_x &= u_x\cdot(x+ct,t) \\ v_{xx} &= u_{xx}(x+ct,t) \end{aligned}\right] \quad \text{Since } u_t + cu_x = u_{xx}$$

So $v_t = v_{xx}$.

(b) $u = \ln v \qquad u_t = \dfrac{v_t}{v}$

$$u_x = \frac{v_x}{v}, \qquad u_{xx} = \frac{v v_{xx} - v_x^2}{v^2}$$

So $u_t = u_{xx} + (u_x)^2 \Longrightarrow \dfrac{v_t}{v} = \dfrac{v\, v_{xx} - \cancel{v_x^2}}{v^2} + \cancel{\left(\dfrac{v_x}{v}\right)^2}$

$\Rightarrow \boxed{v_t = v_{xx}}$

2. Let $\{\mathbf{W}_1, \mathbf{W}_2, \ldots \mathbf{W}_n, \ldots\}$ be a complete orthogonal basis – can be finitely or infinitely many – of a vector space. Let $\mathbf{U}$ and $\mathbf{V}$ be two vectors from the same vector space. Prove that

$$\langle \mathbf{U}, \mathbf{V} \rangle = \frac{\langle \mathbf{U}, \mathbf{W}_1 \rangle \langle \mathbf{V}, \mathbf{W}_1 \rangle}{\langle \mathbf{W}_1, \mathbf{W}_1 \rangle} + \frac{\langle \mathbf{U}, \mathbf{W}_2 \rangle \langle \mathbf{V}, \mathbf{W}_2 \rangle}{\langle \mathbf{W}_2, \mathbf{W}_2 \rangle} + \cdots + \frac{\langle \mathbf{U}, \mathbf{W}_n \rangle \langle \mathbf{V}, \mathbf{W}_n \rangle}{\langle \mathbf{W}_n, \mathbf{W}_n \rangle} + \cdots$$

$$\left\{ \begin{aligned} U &= \frac{\langle U, W_1 \rangle}{\langle W_1, W_1 \rangle} W_1 + \frac{\langle U, W_2 \rangle}{\langle W_2, W_2 \rangle} W_2 + \cdots \\ V &= \frac{\langle V, W_1 \rangle}{\langle W_1, W_1 \rangle} W_1 + \frac{\langle V, W_2 \rangle}{\langle W_2, W_2 \rangle} W_2 + \cdots \end{aligned} \right\}$$

(Note : $W_1, W_2, \ldots$ are orthogonal)

$$\langle U, V \rangle = \left\langle \sum_{i=1}^{\infty} \frac{\langle U, W_i \rangle}{\langle W_i, W_i \rangle} W_i, \sum_{j=1}^{\infty} \frac{\langle V, W_j \rangle}{\langle W_j, W_j \rangle} W_j \right\rangle$$

$$= \sum_{i,j} \frac{\langle U, W_i \rangle}{\langle W_i, W_i \rangle} \frac{\langle V, W_j \rangle}{\langle W_j, W_j \rangle} \langle W_i, W_j \rangle$$

$$= \sum_{i} \frac{\langle W, W_i \rangle \langle V, W_i \rangle}{\langle W_i, W_i \rangle}$$

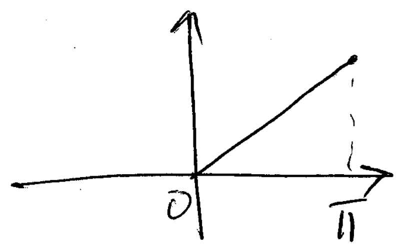
Note:

$(\langle W_i, W_j \rangle = 0 \text{ if } i \neq j)$

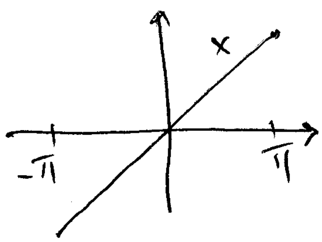
3. Consider the function $f(x) = x$ for $0 \leq x \leq \pi$.

(a) Express $f(x)$ as a sum of series consisting of only $\sin nx$'s.

(b) Express $f(x)$ as a sum of series consisting of only $\cos nx$'s.



(a) Consider odd extension of $f(x)$ → f^{odd}



$$f^{odd}(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\left[-\frac{x \cos nx}{n}\Big|_0^{\pi} + \int_0^{\pi} \frac{\cos nx \, dx}{n} \right]$$

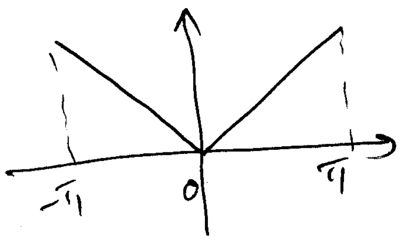
$$= -\frac{2\cos(n\pi)}{n} + \cancel{\frac{2}{\pi}\frac{\sin nx}{n^2}\Big|_0^{\pi}} \to 0$$

$$= -\frac{2(-1)^n}{n} = \frac{2(-1)^{n+1}}{n}$$

$$f^{odd}(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$

This is a scrap paper.

(b) Consider even extension of $f(x)$ $\quad f^{even}$



$$f^{even}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f^{even}(x) \cos nx \, dx = \frac{2}{\pi}\int_0^{\pi} x \cos nx \, dx$$

$$= \frac{2}{\pi}\left[\cancelto{0}{\frac{x \sin nx}{n}\Big|_0^{\pi}} - \int_0^{\pi} \frac{\sin nx}{n} \, dx \right]$$

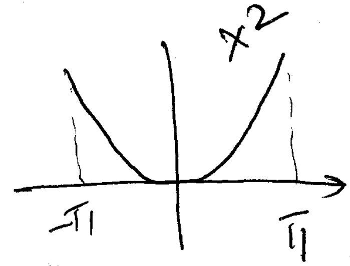
$$= \frac{2}{\pi}\left[+ \frac{1}{n^2} \cos nx \Big|_0^{\pi} \right]$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi - 1}{n^2} \right] = \frac{2}{\pi n^2}\left((-1)^n - 1\right) \quad (n \neq 0)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x \, dx = \frac{\cancel{2}}{\pi}\frac{x^2}{\cancel{2}}\Big|_0^{\pi} = \pi.$$

$$\boxed{f^{even}(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left((-1)^n - 1\right) \cos nx}$$

4. You are given the following Fourier series representation:



$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx, \qquad -\pi \le x \le \pi$$

Prove that

(a) $x^3 - \pi^2 x = 12\sum_{n=1}^{\infty}\frac{(-1)^n \sin nx}{n^3}$ for $-\pi \le x \le \pi$.

(b) $x^4 - 2\pi^2x^2 = 48\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos nx}{n^4} - \frac{7\pi^4}{15}$ for $-\pi \le x \le \pi$.

(c) $\pi^4 = \frac{720}{7}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}$

(a) integrate $\Rightarrow$ $\frac{x^3}{3} = \frac{\pi^2 x}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n \sin nx}{n^3} + C$

Set $x=0 \Rightarrow C=0$

So $x^3 - \pi^2 x = 12\sum_{n=1}^{\infty}\frac{(-1)^n \sin nx}{n^3}$

(b) integrate again $\Rightarrow$

?

$$\frac{x^4}{4} - \frac{\pi^2x^2}{2} = -12\sum_{n=1}^{\infty}\frac{(-1)^n \cos nx}{n^4} + C$$

$$\int_{-\pi}^{\pi}\left(\frac{x^4}{4} - \frac{\pi^2x^2}{2}\right)dx = \int_{-\pi}^{\pi}\left(\overbrace{\cancel{-12\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^4}}}^{0} + c\right)dx$$

$$\cancel{2}\times\left[\frac{\pi^5}{20} - \frac{\pi^5}{6}\right] = C\times\cancel{2}\pi \qquad C = \frac{\pi^4}{20} - \frac{\pi^4}{6}$$

$$= \frac{-7\pi^4}{60}$$

This is a scrap paper.

So $$x^4 - 2\pi^2 x^2 = 48 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cos nx}{n^4} - \frac{7\pi^4}{15}$$

(c) Set $x = 0$

$$\frac{7\pi^4}{15} = 48 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$$

$$\boxed{\pi^4 = \frac{720}{7} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}}$$