1. You are given the following facts:

$$\begin{pmatrix} -5 & 2 \\ 2 & -8 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = -4\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -5 & 2 \\ 2 & -8 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = -9\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Solve the following systems:

(a)

$$\begin{aligned} x_t(t) &= -5x(t) + 2y(t), \\ y_t(t) &= 2x(t) - 8y(t) \end{aligned}$$

such that $x(0) = 4$ and $y(t) = 1$.

(b)

$$\begin{aligned} x_{tt}(t) &= -5x(t) + 2y(t), \\ y_{tt}(t) &= 2x(t) - 8y(t) \end{aligned}$$

such that $x(0) = 4$, $y(t) = 1$, $x_t(0) = -1$ and $y_t(0) = 2$.

(a) $\begin{pmatrix} x \\ y \end{pmatrix}_t = \begin{bmatrix} -5 & 2 \\ 2 & -8 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ (Heat equation type)

$$\begin{pmatrix} x \\ y \end{pmatrix} = a_1 e^{\lambda_1 t} V_1 + a_2 e^{\lambda_2 t} V_2$$

$$\boxed{\begin{aligned} \lambda_1 &= -4 \\ \lambda_2 &= -9 \end{aligned}}$$

$$= a_1 e^{-4t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + a_2 e^{-9t}\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

$\boxed{t=0}$ $\begin{pmatrix} 4 \\ 1 \end{pmatrix} = a_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Note V_1, V_2 are orthogonal !

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$$a_1 = \frac{\left\langle \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle} = \frac{9}{5}$$

$$a_2 = \frac{\left\langle \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\rangle} = \frac{-2}{5}$$

$$\boxed{X(t) = \frac{9}{5} e^{-4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{2}{5} e^{-9t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}}$$

$X_{tt} = AX$ (Wave equation type)

$$X(t) = \left(a_1 \cos\sqrt{-\lambda_1}\, t + b_1 \sin\sqrt{-\lambda_1}\, t\right) V_1 + \left(a_2 \cos\sqrt{-\lambda_2}\, t + b_2 \sin\sqrt{-\lambda_2}\, t\right) V_2$$

$$\boxed{\begin{matrix} \lambda_1 = -4 \\ \lambda_2 = -9 \end{matrix}}$$

$\underline{t=0}$ $\Rightarrow \begin{pmatrix} 4 \\ 1 \end{pmatrix} = a_1 V_1 + a_2 V_2 \Rightarrow a_1 = 9/5,\ a_2 = -2/5$

$$X'(t) = \left(-\sqrt{-\lambda_1}\, a_1 \sin\sqrt{-\lambda_1}\, t + \sqrt{-\lambda_1}\, b_1 \cos\sqrt{-\lambda_1}\, t\right) V_1 + \left(-\sqrt{-\lambda_2}\, a_2 \sin\sqrt{-\lambda_2}\, t + \sqrt{-\lambda_2}\, b_2 \cos\sqrt{-\lambda_1}\, t\right) V_2$$

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$$t=0 \Rightarrow \begin{pmatrix}-1\\2\end{pmatrix} = 2b_1 V_1 + 3b_2 V_2.$$

$$\text{So} \quad 2b_1 = \frac{\begin{pmatrix}-1\\2\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}}{\begin{pmatrix}2\\1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}} = 0. \quad \boxed{b_1 = 0}$$

$$3b_2 = \frac{\begin{pmatrix}-1\\2\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix}}{\begin{pmatrix}-1\\2\end{pmatrix}\begin{pmatrix}-1\\2\end{pmatrix}} = 1 \Rightarrow b_2 = 1/3$$

$$\text{So} \quad \boxed{X(t) = \left(\frac{9}{5}\cos 2t\right)\begin{pmatrix}2\\1\end{pmatrix} + \left(-\frac{2}{5}\cos 3t + \frac{1}{3}\sin 3t\right)\begin{pmatrix}-1\\2\end{pmatrix}}$$

2. Let $f(x,y)$ be a 2π-periodic function in both variables. Consider the following double Fourier series expansion of $f(x,y)$ for $(x,y) \in (-\pi,\pi)\times(-\pi,\pi)$:

$$f(x,y) = \frac{c_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx + \sum_{n=1}^{\infty} a'_n \cos ny + b'_n \sin ny$$
$$+ \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} c_{kl} \cos kx \cos ly + d_{kl} \cos kx \sin ly + e_{kl} \sin kx \sin ly$$

(a) Derive an expression in terms of f for each of the coefficients $a_n, a'_n, b_n, b'_n, c_0, c_{kl}, d_{kl}, e_{kl}$.

(b) Derive an expression for $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f^2(x,y)\,dx\,dy$ in terms of the coefficients.

$$\|1\|^2 = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} 1\,dx\,dy = 4\pi^2$$

$$\|\cos nx\|^2 = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \cos^2 nx\,dx\,dy =$$

$$= 2\pi \underbrace{\int_{-\pi}^{\pi} \cos^2 nx\,dx}_{\shortparallel} = \boxed{2\pi^2}$$

$$2\int_0^{\pi} \cos^2 nx\,dx$$

$$= \int_0^{\pi} (\cancel{\cos 2nx} + 1)\,dx = \pi$$

Similarly, $\|\sin nx\|^2 = 2\pi^2$

$$\|\cos kx \sin ly\|^2 = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \cos^2 kx \sin^2 ly\,dx\,dy$$

$$= \int_{-\pi}^{\pi} \cos^2 kx\,dx \int_{-\pi}^{\pi} \sin^2 ly\,dy = \boxed{\pi^2}$$

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So $$\frac{c_0}{2} = \frac{\langle f, 1\rangle}{\|1\|^2} = \frac{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\,dx\,dy}{4\pi^2}$$

$$c_0 = \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\,dx\,dy.$$

$$a_n = \frac{\langle f, \cos nx\rangle}{\|\cos nx\|^2} = \frac{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\cos nx\,dx\,dy}{2\pi^2}$$

$$b_n = \frac{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\sin nx\,dx\,dy}{2\pi^2}$$

$$a_n' = \frac{\iint f(x,y)\cos ny\,dx\,dy}{2\pi^2}$$

$$b_n' = \frac{\iint f(x,y)\sin ny\,dx\,dy}{2\pi^2}$$

$$c_{kl} = \frac{\iint f(x,y)\cos kx\cos ly\,dx\,dy}{\pi^2}$$

$$d_{kl} = \frac{\iint f(x,y)\cos kx\sin ly\,dx\,dy}{\pi^2}$$

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$$e_{kl} = \frac{\iint f(x,y) \sin kx \sin ly \, dx\,dy}{\pi^2}$$

(b) $\|f\|^2 =$ (Pythagorean / Parsevals Identity)

$$= \left(\frac{c_0}{2}\right)^2 \underbrace{\|1\|^2}_{4\pi^2} + \sum_{n=1}^{\infty} a_n^2 \overbrace{\|\cos nx\|^2}^{2\pi} + b_n^2 \overbrace{\|\sin ny\|^2}^{2\pi}$$

$$+ \sum_{n=1}^{\infty} a_n'^2 \|\cos ny\|^2 + b_n'^2 \|\sin ny\|^2$$

$$+ \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} c_{kl} \overbrace{\|\cos kx \cos ly\|^2}^{\pi^2} + d_{kl} \|\cos kx \sin ly\|^2 + e_{kl} \|\sin kx \sin ly\|^2$$

$$= \boxed{\pi^2 c_0^2 + 2\pi^2 \sum_{n=1}^{\infty} (a_n^2 + b_n^2 + a_n'^2 + b_n'^2) + \pi^2 \sum_{k,l} (c_{kl}^2 + d_{kl}^2 + e_{kl}^2)}$$

3. (a) Consider the following equation:

$$\begin{aligned} \Delta u(x,y) &= \rho(x,y), \quad (x,y) \in \Omega \\ \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} &= f \end{aligned}$$

where Ω is a two dimension domain (such as a square) in R^2 and $\frac{\partial}{\partial n}$ denotes the outward unit normal derivative at $\partial\Omega$.

Find the condition on ρ and f such that the above equation has a solution.

(b) Consider the following equation:

$$\begin{aligned} u_{xx}(x) &= x^3 - x \quad x \in (0,1) \\ u_x(0) = a, &\qquad u_x(1) = b. \end{aligned}$$

Find a condition for a and b such that the above equation has a solution. Furthermore, using the condition you have just found, solve the equation explicitly.

(Note: It is *not* necessary to use eigenvalues and eigenfunctions to solve this one-dimensional problem.)

(a).

$$\Delta u = \rho$$

$$\iint_\Omega \Delta u = \iint_\Omega \rho$$

$$\iint_\Omega \operatorname{div} \nabla u = \iint_\Omega \rho$$

$$\int_{\partial\Omega} \langle \nabla u, n \rangle = \iint_\Omega \rho$$

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} = \iint_\Omega \rho$$

So

$$\boxed{\int_{\partial\Omega} f = \iint_\Omega \rho}$$

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$$u_{xx} = x^3 - x \qquad x \in (0, 1)$$

$$u_x(0) = a, \quad u_x(1) = b$$

$$\int_0^1 u_{xx}\,dx = \int_0^1 (x^3 - x)\,dx$$

$$u_x(1) - u_x(0) = 1/4 - 1/2 = -1/4$$

So $\boxed{b - a = -1/4}$

To solve $u_{xx} = x^3 - x$

$$\Rightarrow \quad u_x = \frac{x^4}{4} - \frac{x^2}{2} + C$$

$x = 0 \Rightarrow \quad a = C$

So $\quad u_x(x) = \frac{x^4}{4} - \frac{x^2}{2} + a$.

($x = 1$ is automatically satisfied ! by the compatibility condition.)

$$b = \frac{1}{4} - \frac{1}{2} + a$$

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$$u_x = \frac{x^4}{4} - \frac{x^2}{2} + a.$$

$$\boxed{u(x) = \frac{x^5}{20} - \frac{x^3}{6} + ax + c}$$

(free!)

4. (a) Consider the operator $Lf = \partial_x\left((x^5+1)\partial_x f\right)$ for $x \in (0,1)$ subject to the Neumann boundary condition: $f_x(0) = f_x(1) = 0$. Show the following fact:

All the eigenvalues of L are *negative* except one *zero* eigenvalue.

In addition, find the corresponding eigenfunction corresponding to the zero eigenvalue.

(b) With the above L, consider the differential equation ~~$u_t = Lu$ with Neumann~~ boundary condition $u_x(0,t) = u_x(1,t) = 0$ and initial value $u(x,0) = x^3 - x$.

Find $\lim_{t\to+\infty} u(x,t)$.

$-f(r)$

(a) $Lf = \lambda f.$

$$\int_0^1 f\,\partial_x\left((x^5+1) f_x\right) dx = \lambda \int_0^1 f^2 dx$$

$$\cancel{f(x^5+1) f_x \Big|_0^1} - \underbrace{\int_0^1 (x^5+1) f_x^2\, dx}_{-ve} = \lambda \underbrace{\int_0^1 f^2 dx}_{+ve}$$

$(f_x(1) = f_x(0) = 0)$

So $\boxed{\lambda \le 0}$

if $\boxed{\lambda = 0} \Rightarrow \partial_x\left((x^5+1) f_x\right) = 0$

$$(x^5+1) f_x = \text{Const} = c$$

$$f_x = \frac{c}{x^5+1}$$

$x=0 \Rightarrow f_x = 0 \Rightarrow c = 0$ So $f_x = 0$

for example

$\Rightarrow \boxed{f = \text{constant} = 1}$

Sturm-Liouville Op.

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$$\text{(b)} \qquad u_t = Lu = \partial_x\left((x^2+1)\,\partial_x f\right)$$

$$\text{So} \quad u(x,t) = c_0 e^{\lambda_0 t}(1) + \sum_{n=1}^{\infty} c_n e^{\lambda_n t}\varphi_n(x)$$

$$u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n e^{\lambda_n t}\varphi_n(x)$$

$\boxed{\lambda_0 = 0}$ $\qquad \lambda_n < 0$!

So as $t \to +\infty$

$$u(x,t) \longrightarrow c_0$$

$\boxed{\text{What is } c_0?}$ $\quad \cancel{c_0 = \frac{\langle u, 1\rangle}{\langle 1, 1\rangle} = \int_0^1 u(x,1}$

Let $t=0$ $\quad f(x) = c_0 + \sum_{n=1}^{\infty} c_n\varphi_n(x)$

<u>Note</u> $\{1, \varphi_n(x) \dots\}$ are $\perp$ fcts $\quad x^3+x \quad -1/4$

($\because$ Sturm-Liouville Op)

$$\text{So} \quad c_0 = \frac{\langle f, 1\rangle}{\langle 1, 1\rangle} = \frac{\int_0^1 f(x)\,dx}{\int_0^1 dx} = \boxed{\int_0^1 f(x)\,dx}$$