MA 520: Boundary Value Problems of Differential Equations
Spring 2020, Midterm Exam

Instructor: Yip

- This test booklet has FIVE QUESTIONS, totaling 100 points for the whole test. You have 75 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, closed note, with no electronic devices**.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.

Some Useful Formula

1. The eigenvalues and eigenfunctions for ∂_{xx} with homogeneous Dirichlet boundary conditions on $(0, \pi)$ are given by $\lambda_n = -n^2$ and $\phi_n(x) = \sin(nx)$ for $n = 0, 1, 2, \ldots$.
2. Solution of of $\dot{c}(t) = ac(t) + b(t)$ with initial condition $c(0) = c_0$ is given by

$$c(t) = c_0 e^{at} + \int_0^t e^{a(t-s)} b(s)\, ds.$$

Name: Answer Key (Major:)

Question	Score
1.(20 pts)	
2.(20 pts)	
3.(20 pts)	
4.(20 pts)	
5.(20 pts)	
Total (100 pts)	

1. Find the Fourier series of the following functions which are defined for $-\pi < x < \pi$:

   (a) x;

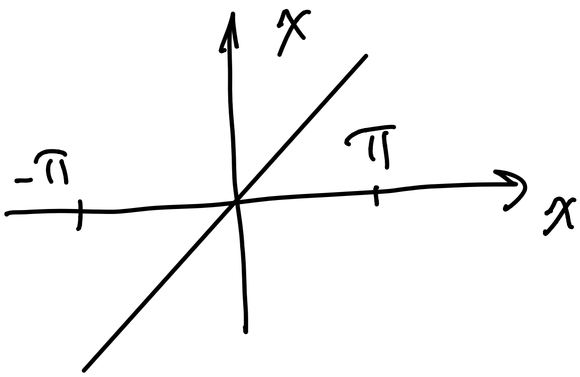
   (b) x^2;

   (c) x^3;

   (d) x^4.

   You should simplify all your constants as much as possible.

(a)



$$x = \sum_{n=1}^{\infty} b_n \sin nx \qquad -\pi < x < \pi$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\left[-\int x \, d\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{x\cos nx}{n}\Big|_{\cancel{0}}^{\pi} + \cancel{\int_0^{\pi} \frac{\cos nx}{n} dx}\right]$$

$$= -\frac{2}{n}\cos n\pi = \frac{2}{n}(-1)^{n+1}$$

$$\boxed{x = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin(nx)}$$

(F. p.26 #1)

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(b) $$x^2 = \int 2x\,dx + C$$

$$= 4\sum_{n=1}^{\infty} \int \frac{(-1)^{n+1}}{n} \sin nx\,dx + C$$

$$= C + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

$$\int_0^{\pi} x^2\,dx = \int_0^{\pi} \left(C + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cos nx \right) dx$$

$$\frac{\pi^3}{3} = \pi C + 0 \implies C = \frac{\pi^2}{3}$$

$$\boxed{x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx}$$

(F. p.28 #16)

(c) $$x^3 = \int 3x^2\,dx$$

$$= \int \left(\pi^2 + \sum_{n=1}^{\infty} \frac{12(-1)^n}{n^2} \cos nx \right) dx$$

$$= C + \pi^2 x + \sum_{n=1}^{\infty} \frac{12(-1)^n}{n^3} \sin nx$$

Set $x=0 \Longrightarrow C=0$

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$$x^3 = \pi^2 x + \sum_{n=1}^{\infty} \frac{12(-1)^n}{n^3} \sin nx$$

$$= \pi^2 \left[\sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx \right] + \sum_{n=1}^{\infty} \frac{12(-1)^n}{n^3} \sin nx$$

$$\boxed{x^3 = \sum_{n=1}^{\infty} \left(\frac{2\pi^2(-1)^{n+1}}{n} + \frac{12(-1)^n}{n^3} \right) \sin nx}$$

(d) $x^4 = \int 4x^3 \, dx$

$$= C + \int \sum_{n=1}^{\infty} \left(\frac{8\pi^2(-1)^{n+1}}{n} + \frac{48(-1)^n}{n^3} \right) \sin nx \, dx$$

$$= C + \sum_{n=1}^{\infty} \left(\frac{8\pi^2(-1)^n}{n^2} + \frac{48(-1)^{n+1}}{n^4} \right) \cos nx$$

$$\int_0^{\pi} x^4 dx = \int_0^{\pi} C \, dx + \underbrace{\cdots}_{\to 0} \Rightarrow C = \frac{\pi^5}{5}$$

$$\boxed{x^4 = \frac{\pi^5}{5} + \sum_{n=1}^{\infty} \left(\frac{8\pi^3(-1)^n}{n^2} + \frac{48(-1)^{n+1}}{n^4} \right) \cos nx}$$

[F. p.86 #3]

2. Let D be the unit disk $\{x^2 + y^2 \le 1\}$ in $\mathbb{R}^2$. Consider the inner product on the space of L^2-functions defined on D as $\langle f, g \rangle = \iint_D f(x,y)\overline{g(x,y)}\,dxdy$. Let further $f_n(x,y) = (x+iy)^n$ for $n = 0, 1, 2, \ldots$.

   (a) Show that $\{f_n\}_{n=0}^{\infty}$ is an orthogonal set. Find also $\|f_n\|$.
   (Hint: use polar coordinates $x + iy = re^{i\theta}$ and the formula $dxdy = rdrd\theta$.)

   (b) Let $f(z) = a_0 + a_1 z + \cdots a_k z^k$ and $g(z) = b_0 + b_1 z + \cdots b_k z^k$ where $z = x + iy$.
   Find $\|f\|, \|g\|, \langle f, g \rangle$.

(a) $$\langle f_n, f_m \rangle_D = \int_0^1 \int_0^{2\pi} (x+iy)^n \overline{(x+iy)}^m \, r \, dr \, d\theta$$

$$= \int_0^1 \int_0^{2\pi} (re^{i\theta})^n (re^{-im\theta}) \, r dr \, d\theta$$

$$= \int_0^1 \int_0^{2\pi} r^{n+m+1} e^{i(n-m)\theta} \, d\theta \, dr$$

$$= \int_0^1 r^{n+m+1} dr \underbrace{\int_0^{2\pi} e^{i(n-m)\theta} d\theta}_{\begin{cases} = 0 & \text{if } n \ne m \\ 2\pi & \text{if } n = m \end{cases}}$$

for $n=m$,

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$$\|f_n\|^2 = \langle f_n, f_n \rangle = \left(\int_0^1 r^{2n+1}\,dr\right) 2\pi$$

$$= \frac{2\pi}{2n+2} = \frac{\pi}{n+1}$$

$$\boxed{\|f_n\| = \sqrt{\frac{\pi}{n+1}}}$$

(b) $f(z) = a_0 + a_1 z + \cdots + a_k z^k$

$g(z) = b_0 + b_1 z + \cdots + b_k z^k$

$\langle f, g \rangle_D$

$= a_0 \overline{b_0} \langle 1, 1 \rangle + \cdots + \langle a_k, b_k \rangle \langle z_k, z_k \rangle$

$$\boxed{= a_0\overline{b_0}\frac{\pi}{1} + a_1\overline{b_1}\frac{\pi}{2} + \cdots a_k\overline{b_k}\frac{\pi}{k+1} \quad \left(= \sum_{n=0}^{k} a_n \overline{b_n} \frac{\pi}{n+1}\right)}$$

3. Consider the following one dimensional heat equation:

$$\begin{aligned} u_t &= u_{xx} + x, \quad x \in (0,\pi) \\ \boxed{u(0,t) = 0, \quad u(\pi,t) = 0,} \\ u(x,0) &= 0. \end{aligned}$$

Find $u(x,t)$ and $\lim_{t\to+\infty} u(x,t)$.

Method 1

$$x = \sum_{n=1}^{\infty} b_n \underline{\sin nx}$$

$$\left(\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx \\ &= \frac{2}{\pi}\left[-\frac{x\cos nx}{n}\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi} \cos nx \, dx \right] \\ &= \frac{2(-1)^{n+1}}{n} \end{aligned} \right)$$

$$= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx \quad \text{(See also prob. \#1)}$$

Let $\displaystyle u(x,t) = \sum_{n=1}^{\infty} c_n(t) \sin nx$

$$\Downarrow$$

$$u_t = u_{xx} + x$$

$$\Longrightarrow \quad \dot{c}_n(t) = -n^2 c_n(t) + \frac{2(-1)^{n+1}}{n}, \quad n \geq 1$$

$$C_n(t) = \cancel{C_n(0)}e^{-n^2t} + e^{-n^2t}\int_0^t e^{n^2s}\frac{2(-1)^{n+1}}{n}\,ds$$

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0

$$= e^{-n^2t}\frac{2(-1)^{n+1}}{n}\left(\frac{e^{n^2s}}{n^2}\Big|_0^t\right)$$

$$= e^{-n^2t}\frac{2(-1)^{n+1}}{n^3}\left(e^{n^2t}-1\right)$$

$$= \frac{2(-1)^{n+1}}{n^3}\left(1-e^{-n^2t}\right)$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n^3}\left(1-e^{-n^2t}\right)\sin nx}$$

as $t \to +\infty$, $e^{-n^2t} \longrightarrow 0$

$$\boxed{u(x,t) \longrightarrow \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n^3}\sin nx}$$

Method 2 $\boxed{\text{Make use of steady state}}$

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Find $\tilde{u}$ s.t. $\tilde{u}_{xx} + x = 0$

$$\tilde{u}_{xx} = -x \Longrightarrow \tilde{u}_x = -\frac{x^2}{2} + a$$

$$\Longrightarrow \tilde{u} = -\frac{x^3}{6} + ax + b$$

$\tilde{u}(0) = 0 \Rightarrow b = 0$

$\tilde{u}(\pi) = 0 \Rightarrow a = \frac{\pi^2}{6}$

Hence $\boxed{\tilde{u} = -\frac{x^3}{6} + \frac{\pi^2 x}{6}}$

Set $u = \tilde{u} + g$

Then $\boxed{g_t = g_{xx}, \quad g(0,t) = g(\pi,t), \quad g(x,0) = -\tilde{u}}$

$$g(x,t) = \sum_{n=1}^{\infty} e^{-n^2 t} \underline{\underline{C_n(0)}} \sin nx$$

$$C_n(0) = \frac{2}{\pi} \int_0^{\pi} -\tilde{u}(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(-\frac{\pi^2 x}{6} + \frac{x^3}{6}\right)\sin nx \, dx$$

(use prob. #1)

$$= -\frac{\pi^2}{6}\cancel{\frac{2(-1)^{n+1}}{n}} + \frac{1}{6}\left(\cancel{-\frac{2\pi^2}{n}} + \frac{12}{n^3}\right)(-1)^n$$

$$= \frac{2}{n^3}(-1)^n$$

Hence $u(x,t) = \tilde{u}(x) + g(x,t)$

$$= \left(\frac{x\pi^2}{6} - \frac{x^3}{6}\right) + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^3} e^{-n^2 t} \sin nx$$

4. You are given the following information:

$$\begin{pmatrix} -5 & 2 \\ 2 & -8 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = -4\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -5 & 2 \\ 2 & -8 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = -9\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Solve the following system of differential equations:

$$\begin{aligned} x_t(t) &= -5x(t) + 2y(t) + 4 - t, \\ y_t(t) &= 2x(t) - 8y(t) + 2 + 2t \end{aligned}$$

such that $x(0) = 4$ and $y(t) = 1$.

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 & 2 \\ 2 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4-t \\ 2+2t \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1(t)\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2(t)\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 4-t \\ 2+2t \end{pmatrix} = b_1(t)\begin{pmatrix} 2 \\ 1 \end{pmatrix} + b_2(t)\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

orthogonal

$$b_1(t) = \frac{\left\langle \begin{pmatrix} 4-t \\ 2+2t \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle}{5} = \frac{8-2t+2+2t}{5} = 2$$

$$b_2(t) = \frac{\left\langle \begin{pmatrix} 4-t \\ 2+2t \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\rangle}{5} = \frac{-4+t+4+4t}{5} = t$$

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$$\boxed{\dot{C}_1(t) = -4C_1(t) + 2}$$

$$C_1(t) = C_1(0)e^{-4t} + e^{-4t}\int_0^t e^{4s}2\,ds$$

$$= C_1(0)e^{-4t} + e^{-4t}\frac{1}{2}(e^{4t}-1)$$

$$= C_1(0)e^{-4t} + \frac{1}{2} - \frac{1}{2}e^{-4t}$$

$$\boxed{\dot{C}_2(t) = -9C_2(t) + t}$$

$$C_2(t) = C_2(0)e^{-9t} + e^{-9t}\int_0^t e^{9s}s\,ds$$

$$= C_2(0)e^{-9t} + e^{-9t}\left[\frac{se^{9s}}{9}\Big|_0^t - \frac{1}{9}\int_0^t e^{9s}\,ds\right]$$

$$= C_2(0)e^{-9t} + e^{-9t}\left[\frac{te^{9t}}{9} - \frac{1}{81}e^{9t} + \frac{1}{81}\right]$$

$$= C_2(0)e^{-9t} + \frac{t}{9} - \frac{1}{81} + \frac{e^{-9t}}{81}$$

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$$C_1(0) = \frac{\left\langle \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle}{5} = \frac{9}{5}$$

$$C_2(0) = \frac{\left\langle \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\rangle}{5} = -\frac{2}{5}$$

$$\boxed{\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= C_1(t)\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2(t)\begin{pmatrix} -1 \\ 2 \end{pmatrix} \\ &= \left(\frac{9}{5}e^{-4t} + \frac{1}{2} - \frac{1}{2}e^{-4t}\right)\begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &\quad + \left(-\frac{2}{5}e^{-9t} + \frac{t}{9} - \frac{1}{81} + \frac{e^{-9t}}{81}\right)\begin{pmatrix} -1 \\ 2 \end{pmatrix} \end{aligned}}$$

5. Consider the one dimensional Laplace operator $\mathcal{L}f = D\partial_x^2 f$ on the space of (L^2-)functions defined on the interval $(0, L)$ *endowed with boundary conditions* $f(0) = 0$ *and* $f(L) = 0$. Let $\langle f,\, g\rangle = \int_0^L f(x)g(x)\,dx$ be the standard inner product.

   (a) Prove that $\langle \mathcal{L}f,\, g\rangle = \langle f,\, \mathcal{L}g\rangle$, i.e. $\mathcal{L}$ is symmetric in the context of linear algebra.

   (b) Prove that $\langle \mathcal{L}f,\, f\rangle \leq 0$, i.e. $\mathcal{L}$ is negative (semi-)definite in the context of linear algebra.

   (c) Let ϕ be an eigenfunction of $\mathcal{L}$ with eigenvalue λ, i.e. $\mathcal{L}\phi = \lambda\phi$ (with $\phi \not\equiv 0$). Prove that λ must be negative.

   (d) Let ϕ and ψ be functions with distinct eigenvalues $\lambda \neq \mu$: $\mathcal{L}\phi = \lambda\phi$ and $\mathcal{L}\psi = \mu\psi$. Prove that $\phi \perp \psi$, i.e. $\langle \phi,\, \psi\rangle = 0$.

(Remarks:
(i) Hint for (a) and (b): use integration by parts. Beware of boundary conditions.
(ii) Hint for (d): apply (a) with ϕ and ψ.
(iii) For (c) and (d), you need to prove the statements *without* using any explicit formula about the eigenvalues and eigenfunctions, for example, those given in the first page.)

(a) $$\underbrace{\int_0^L f_{xx}\, g\,dx}_{=\langle \mathcal{L}f,\, g\rangle} = \cancel{f_x g}\Big|_0^L - \int_0^L f_x g_x\,dx$$

$g(L) = g(0) = 0$

$$= -\cancel{f g_x}\Big|_0^L + \underbrace{\int_0^L f\, g_{xx}\,dx}_{=\langle f,\, \mathcal{L}g\rangle}$$

$f(L) = f(0) = 0$

(b) $$\langle \mathcal{L}f, f\rangle = \int_0^L f_{xx} f\,dx = \cancel{f_x f}\Big|_0^L - \int_0^L f_x f_x\,dx$$

0

$$= -\int_0^L f_x^2\,dx \leq 0$$

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(c) $\quad f_{xx} = \lambda f$

$$\Rightarrow \langle f_{xx}, f \rangle = \lambda \langle f, f \rangle$$

i.e. $$\int_0^L f_{xx}\, f\, dx = \lambda \int_0^2 f^2\, dx$$

L.H.S. by (b) $= -\int_0^L f_x^2\, dx$

i.e. $$\underset{-ve}{-} \underbrace{\int_0^L f_x^2\, dx}_{+ve} = \lambda \underbrace{\int_0^L f^2\, dx}_{+ve}$$

$$\Rightarrow \lambda \leq 0$$

$\boxed{\text{If } \lambda = 0}$ $\Rightarrow f_{xx} = 0, \Rightarrow f(x) = Ax + B$

$f(0) = 0 \Rightarrow B = 0$

$f(L) = 0 \Rightarrow A = 0$

i.e. $\boxed{f(x) \equiv 0. \text{ Not possible for eigenfunction.}}$

d) $\varphi_{xx} = \lambda\varphi, \quad \psi_{xx} = \mu\psi$

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$$\langle \varphi_{xx}, \psi \rangle \overset{(a)}{=} \langle \varphi, \psi_{xx} \rangle$$

$$\langle \lambda\varphi, \psi \rangle = \langle \varphi, \mu\psi \rangle$$

$$\lambda \langle \varphi, \psi \rangle = \mu \langle \varphi, \psi \rangle$$

$$\boxed{\lambda \neq \mu \implies \langle \varphi, \psi \rangle = 0}$$