Math520 Spring2023 Midterm 2

Mar 27, 2023. 1:30-2:20pm

Instructions

1. Read each problem carefully, and follow the instructions.

2. Show all of your work and clearly indicate all answers. Justify procedures and calculations except when they are obvious. **Circle** your final answer.

3. Check that your test has 4 problems. Each question will have its point value listed with the problem itself.

4. The use of books, notes, calculator and electronic devices are **not** permitted.

5. The last page is scratch paper. You can write on the back of each page if there is no enough space, but make sure all the procedures and answers must be on the stapled exam paper. **The scratch paper will NOT be turned in or graded.**

6. Make sure you **sign** the honor pledge below.

I, ________________________, pledge that I have neither given nor received any unauthorized assistance on this exam, and I have conducted myself within the guidelines of Purdue's community standard. Moreover, I will not discuss the content of this exam until authorized to do so.

1(25'). Consider the inhomogeneous wave equation with damping

$$\begin{aligned}
&\partial_{tt}u = 5\partial_{xx}u - 4\partial_t u + \cos x, \quad x \in [0, L],\ t > 0;\\
&\partial_x u(0,t) = 0, \quad \partial_x u(L,t) = 0, \quad t > 0;\\
&u(x,0) = f(x), \quad \partial_t u(x,0) = g(x), \quad x \in (0, L).
\end{aligned}$$

(1) Construct a stationary solution $w(x)$ such that $v(x,t) := u(x,t) - w(x)$ satisfies

$$\begin{aligned}
&\partial_{tt}v = 5\partial_{xx}v - 4\partial_t v, \quad x \in [0, L],\ t > 0;\\
&\partial_x v(0,t) = 0, \quad \partial_x v(L,t) = 0, \quad t > 0;\\
&v(x,0) = f(x) - w(x), \quad \partial_t v(x,0) = g(x), \quad x \in (0, L).
\end{aligned}$$

(Hint: Write down equations for w without solving them.)
(2) Suppose you are given the eigenvalues and eigenfunctions $\lambda_n, \phi_n,\ n = 0, 1, \cdots$ for the corresponding Strum-Liouville problem

$$5\phi''(x) + \lambda\phi(x) = 0, \quad \phi'(0) = \phi'(L) = 0,$$

then one can express $v(x,t)$ as $\sum_n c_n(t)\phi_n(x)$. For its coefficients $c_n(t)$, derive the 2nd order ODE.

A(1.1): $w(x)$ satisfies

$$5w''(x) + \cos x = 0, \quad w'(0) = 0,\ w'(L) = 0.$$

A(1.2): Plugging $v(x,t) = \sum_n c_n(t)\phi_n(x)$ into v-equations, we have

$$LHS = \sum_n c_n''(t)\phi_n = RHS = -\sum_n \lambda_n c_n(t)\phi_n(x) - 4\sum_n c_n'(t)\phi_n.$$

Thus the 2nd order ODE for $c_n(t)$ are

$$c_n'' + 4c_n' + \lambda_n c_n = 0.$$

2(25'). (1) Solve (derivations not required) the heat equation with inhomogeneous boundary conditions

$$\begin{aligned} &\partial_t u = \partial_{xx} u, \quad x \in [0, L],\ t > 0; \\ &u(0,t) = 0, \quad u(L,t) = 1, \quad t > 0; \\ &u(x,0) = \sin\frac{\pi x}{L} + \frac{x}{L}, \quad x \in (0, L). \end{aligned}$$

(2) As $t \to +\infty$, what's the stationary solution that $u(x,t)$ will converge to?
Hint: You may directly use the formula

$$\frac{2}{L}\int_0^L \sin\frac{\pi x}{L}\sin\frac{n\pi x}{L}\,\mathrm{d}x = \begin{cases} 1, & n = 1; \\ 0, & n = 2, 3, \cdots. \end{cases}$$

A(2.1):

- Construct $w(x)$ such that $w''(x) = 0$ and $w(0) = 0$, $w(L) = 1$. That gives $w(x) = \frac{x}{L}$.
- $v(x,t) = u(x,t) - w(x)$ solves the heat equation with homo. BC

$$\begin{aligned} &\partial_t v = \partial_{xx} v, \quad x \in [0, L],\ t > 0; \\ &v(0,t) = 0, \quad v(L,t) = 0, \quad t > 0; \\ &v(x,0) = \sin\frac{\pi x}{L}, \quad x \in (0, L). \end{aligned}$$

- Eigenvalues and eigenfunctions for the corresponding Strum-Liouville problem

$$X'' + \lambda X = 0, \quad X(0) = X(L) = 0$$

  are $\lambda_n = (\frac{n\pi}{L})^2$ and $X_n(x) = \sin\frac{n\pi x}{L}$, $n = 1, 2, \cdots$.
- Thus we obtain

$$v(x,t) = \sum_{n=1}^{+\infty} c_n e^{-(\frac{n\pi}{L})^2 t}\sin\frac{n\pi x}{L} = e^{-\frac{\pi^2}{L^2}t}\sin\frac{\pi x}{L},$$

  where $c_n = \frac{2}{L}\int_0^L \sin\frac{\pi x}{L}\sin\frac{n\pi x}{L}\,\mathrm{d}x = \begin{cases} 1, & n = 1; \\ 0, & n = 2, 3, \cdots. \end{cases}$
- In summary, $u(x,t) = v(x,t) + w(x) = e^{-\frac{\pi^2}{L^2}t}\sin\frac{\pi x}{L} + \frac{x}{L}$.

A(2.2): stationary solution is $\frac{x}{L}$.

3(25'). (1) Directly write down (derivations not required) the eigenvalues and eigenfunctions for one-dimensional boundary value problem

$$\begin{aligned} &\partial_{xx}\phi + \mu\phi = 0, \quad x \in (0, a); \\ &\phi(0) = \phi(a) = 0. \end{aligned}$$

(2) Directly write down (derivations not required) the eigenvalues and eigenfunctions for two-dimensional boundary value problem

$$\begin{aligned} &\Delta u + \lambda u = 0, \quad (x, y) \in \Omega := [0, a] \times [0, b]; \\ &u|_{\partial\Omega} = 0. \end{aligned}$$

(3) If the domain Ω is changed to a disk, does one obtain the same eigenfunctions as in (2)? (Yes/No)
If the domain Ω is changed to a disk, are the eigenvalues still nonnegative as in (2)? (Yes/No)

A(3.1): $\mu_n = (\frac{n\pi}{a})^2$ and $\phi_n(x) = \sin\frac{n\pi x}{a}$, $n = 1, 2, \cdots$.
A(3.2): $\lambda_{mn} = (\frac{n\pi}{a})^2 + (\frac{n\pi}{b})^2$ and $\chi_{mn} = \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$, $n = 1, 2\cdots$, $m = 1, 2, \cdots$.
A(3.3): No. Yes.

4(25'). Derive the general solutions for Dirichlet problem in an annulus

$$
\begin{aligned}
&u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad (r,\theta) \in \Omega := [r_1, r_2] \times [0, 2\pi], \\
&u(r_1, \theta) = f(\theta), \quad u(r_2, \theta) = g(\theta), \quad 0 \le \theta \le 2\pi, \\
&u(r, 0) = u(r, 2\pi), \quad r_1 \le r \le r_2.
\end{aligned}
$$

Hint: you are given the formula for the general solutions to Euler equation $r^2R'' + rR' - \lambda R = 0$:

$$
R(r) = \begin{cases} c_1 \log r + c_2, & \text{if } \lambda = 0; \\ c_1 r^{\sqrt{\lambda}} + c_2 r^{-\sqrt{\lambda}}, & \text{if } \lambda > 0. \end{cases}
$$

A(4): See lecture notes [ch4, page 16-17].