MA 520: Boundary Value Problems of Differential Equations
Spring 2020, Final Exam

Instructor: Yip

- This exam booklet has five questions, totaling 100 points. **It is due at noon (12pm), on Wednesday, May 6. It should be scanned or photographed and emailed directly back to me in one file using PDF format.**
- The exam is open book and open note. You are not allowed to ask or collaborate with anyone or search for answers or hints using the internet.
- You can send me email if something is not clear. If your question is sent by 5pm Tuesday, I will try my best to reply before 9pm.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.

Please copy the following sentence and sign:

I have not collaborated with anyone or searched the internet for answers.

Signature:____________________________

Name: Answer Key (Major:)

Question	Score
1.(20 pts)	
2.(20 pts)	
3.(20 pts)	
4.(20 pts)	
5.(20 pts)	
Total (100 pts)	

1. Consider the following differential equation:

$$t^2\ddot{x}(t) + 3t\dot{x}(t) + x(t) = t, \quad \text{for } t > 1 \text{ with initial values } x(1) = 0, \dot{x}(1) = 1.$$

   (a) Find two linearly independent homogeneous solutions $\varphi_1(t)$ and $\varphi_2(t)$.
   (Try $x(t) = t^r$ for some values of r.)

   (b) Find a particular solution $x_p(t)$.

   (c) Find the solution for the above differential equation.

   (Hint: if you need to integrate the time variable, it might be more convenient if you integrate from the value 1 rather then 0.)

2. Find the Fourier transforms of the following functions. Discuss their behaviors as $|\xi| \longrightarrow \infty$, in particular how fast the Fourier transforms converge to zero.

$$f(x) = \begin{cases} 1, & |x| < 1; \\ 0, & |x| > 1. \end{cases}$$

$$g(x) = \begin{cases} 1 - x, & 0 < x < 1; \\ x + 1, & -1 < x < 0; \\ 0, & |x| > 1. \end{cases}$$

$$h(x) = \begin{cases} (x^2 - 1)^2, & |x| < 1; \\ 0, & |x| > 1. \end{cases}$$

3. Consider the domain $\Omega = \{0 < x < a; 0 < y < b\}$ and the following boundary value problem:

$$\begin{aligned} \triangle u &= h(x,y) \text{ in } \Omega \\ u(0,y) = 0 \text{ and } & u(a,y) = f(y) \text{ for } y \in (0,b) \\ -u_y(x,0) = 0 \text{ and } & u_y(x,b) = g(x) \text{ for } x \in (0,a) \end{aligned}$$

   (a) Find the eigenvalues and eigenfunctions for the Laplace operator $\triangle$ with the corresponding (homogeneous) boundary conditions, i.e. find λ and $\varphi(x,y)$ such that $\triangle\varphi = \lambda\varphi$ (with $f, g = 0$)

   (b) If possible, solve the above boundary value problem. If not possible, state clearly if any compatibility condition is needed and then solve it.
   All the constants in your answer should be expressed explicitly in terms of h, f and g.

4. In class, we have shown that the solution of the following one dimensional heat equation on the whole real line,

$$\begin{aligned} u_t(x,t) &= Du_{xx}(x,t), \quad -\infty < x < \infty, \quad t > 0; \\ u(x,0) &= f(x) \end{aligned}$$

is given using the heat kernel $g(x,t) = \dfrac{1}{\sqrt{4\pi D t}} e^{-\frac{x^2}{4Dt}}$:

$$u(x,t) = \int_{-\infty}^{\infty} f(y) g(x-y,t)\, dy$$

For the following equations, find the solution $u(x,t)$ *using the above* g.

(a) $u_t(x,t) = Du_{xx}(x,t) + au_x(x,t) + bu(x,t)$ and $u(x,0) = f(x)$ for $-\infty < x < \infty$, Here a and b are constants.

(b) $u_t(x,t) = Du_{xx}(x,t)$ and $u(x,0) = f(x)$ for $x > 0$ with Neumann boundary condition $u_x(0,t) = 0$.

(c) $u_t(x,t) = Du_{xx}(x,t)$ and $u(x,0) = f(x)$ for $-\pi < x < \pi$ with periodic boundary condition $u(-\pi,t) = u(\pi,t)$. (Or equivalently, you can also consider $u(x,t)$ to be defined for $-\infty < x < \infty$ but it satisifes the condition $u(x,t) = u(x+2\pi,t)$ for all x.

(Hint: for (a), make use of Fourier transform and its properties; for (b) and (c), use the method of images.)

5. Let f be a smooth function defined for $-\infty < x < \infty$. Introduce the following function:

$$g(x) = \sum_{k=-\infty}^{\infty} f(x + 2k\pi).$$

Here "smooth" means that f has as many derivatives as you want. In addition, you can assume that $f(x)$ goes to zero fast enough as $|x| \longrightarrow \infty$ so that the above sum is well defined. For example, $|f(x)| \le C|x|^{-2}$ or $|f(x)| \le Ce^{-|x|}$ for some constant C.

(a) Show that g is a 2π-periodic function, i.e. $g(x+2\pi) = g(x)$.

(b) Consider the Fourier series expansion of g:

$$g(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Express c_n using the *Fourier transform* $\hat{f}$ of f.

(c) Prove the following "interesting" identity:

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) = 2\pi \sum_{n=-\infty}^{\infty} f(2\pi n)$$

(d) By means of the above or otherwise, find the value of the following sum,

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+1}$$

#1 (a) $t^2\ddot{x} + 3t\dot{x} + x = 0$

Let $x(t) = t^r$. Then

$$r(r-1) + 3r + 1 = 0$$

$$r^2 + 2r + 1 = 0 \implies \boxed{r = -1, -1}$$

repeated root

Set $\boxed{\varphi_1(t) = t^{-1}.}$

Let $\underline{\varphi_2(t) = u(t)\varphi_1(t) = u(t)t^{-1}}$

$$t^2\left[\ddot{u}t^{-1} + 2\dot{u}(-1)t^{-2} + \cancel{u(-1)(-2)t^{-3}}\right]$$

$$+ 3t\left[\dot{u}t^{-1} + \cancel{u(-1)t^{-2}}\right] + \cancel{ut^{-1}} = 0$$

$$t\ddot{u} + \dot{u} = 0$$

$$(t\dot{u})^{\cdot} = 0$$

$$\implies t\dot{u} = A \implies \dot{u} = \frac{A}{t}$$

$$\implies u = A\log t + B$$

$$\boxed{\varphi_2(t) = t^{-1}\log t}$$

(b) $\quad t^2\ddot{x} + 3t\dot{x} + x = h(t)$

$$x(t) = A\varphi_1(t) + B\varphi_2(t) + x_p(t)$$

$$x_p(t) = \int_1^t \frac{-\varphi_1(t)\varphi_2(s) + \varphi_2(t)\varphi_1(s)}{W(s)}\left(\frac{h(s)}{s^2}\right)ds$$

$$\left(W(s) = \begin{vmatrix} \varphi_1(s) & \varphi_2(s) \\ \dot{\varphi}_1(s) & \dot{\varphi}_2(s) \end{vmatrix} = \begin{vmatrix} s^{-1} & s^{-1}\log s \\ -s^{-2} & -s^{-2}\log s + s^{-2} \end{vmatrix} \right.$$

$$\left. = -s^{-3}\log s + s^{-3} + s^{-3}\log s = s^{-3} \right)$$

$$= \int_1^t \frac{-t^{-1}\cancel{s^{-1}}\log s + t^{-1}\log t\, \cancel{s^{-1}}}{\cancel{s^{-3}}} \frac{h(s)}{\cancel{s^2}}\, ds$$

$$= \boxed{\int_1^t t^{-1}(\log t - \log s)\, h(s)\, ds}$$

(c) $x_p(t) = \int_1^t t^{-1}(\log t - \log s)\, s\, ds$

$$= (t^{-1}\log t)\left(\frac{t^2-1}{2}\right) - t^{-1}\int_1^t s\log s\, ds$$

$$= \frac{t^{-1}\log t\,(t^2-1)}{2} - t^{-1}\left[\frac{s^2\log s}{2}\Big|_{1}^{t} - \frac{1}{2}\int_1^t s\, ds\right]$$

$$= \frac{(t^{-1}\log t)(\cancel{t^2}-1)}{2} - t^{-1}\left[\cancel{\frac{t^2\log t}{2}} - \frac{1}{4}(t^2-1)\right]$$

$$= \frac{1}{4}(t - t^{-1}) - \frac{t^{-1}\log t}{2}$$

$$= \underbrace{\frac{t}{4}} - \frac{t^{-1}}{4} - \frac{t^2}{2}\log t \qquad \boxed{-\frac{t^{-1}}{4} + \frac{1}{2}t^{-1}\log t + \frac{t}{4}}$$

$$//$$

$$x(t) = At^{-1} + Bt^{-1}\log t + \frac{t}{4}$$

$$x(1) = 0 \implies A + \frac{1}{4} = 0 \implies \underline{A = -\frac{1}{4}}$$

$$\dot{x}(t) = -At^{-2} + B\left[-t^{-2}\log t + t^{-2}\right] + \frac{1}{4}.$$

$$\dot{x}(1) = 1 \implies -A + B + \frac{1}{4} = 1 \implies \underline{B = \frac{1}{2}}$$

#2 (a) $\hat{f}(\xi) = \int_{-1}^{1} e^{-i\xi x}\,dx$

$$= \left.\frac{e^{-i\xi x}}{-i\xi}\right|_{-1}^{1} = \frac{e^{-i\xi} - e^{i\xi}}{i\xi}$$

$$= \boxed{\frac{2}{\xi}\sin(\xi) \xrightarrow{|\xi|\to\infty} 0 \text{ like } O\left(\frac{1}{|\xi|}\right)}$$

(b) $\hat{g}(\xi) = \int_{-1}^{1} g(x)e^{-i\xi x}\,dx$

$$= \int_{-1}^{0}(x+1)e^{-i\xi x}\,dx + \int_{0}^{1}(1-x)e^{-i\xi x}\,dx$$

$$= \underbrace{\int_{-1}^{1} e^{-i\xi x}\,dx}_{\frac{2\sin\xi}{\xi}} + \underbrace{\int_{-1}^{0} xe^{-i\xi x}\,dx}_{\mathrm{I}} - \underbrace{\int_{0}^{1} xe^{-i\xi x}\,dx}_{\mathrm{II}}$$

$$\text{I} = \int x \, d\left(\frac{e^{-i\xi x}}{-i\xi}\right) = \frac{x e^{-i\xi x}}{-i\xi}\Bigg|_{-1}^{0} + \int_{-1}^{0} \frac{e^{-i\xi x}}{+i\xi}\,dx$$

$$= -\frac{e^{i\xi}}{i\xi} - \frac{1}{(i\xi)^2}\left[e^{-i\xi x}\right]_{-1}^{0}$$

$$= -\frac{e^{i\xi}}{i\xi} + \frac{1}{\xi^2}\left[1 - e^{i\xi}\right]$$

$$\text{II} = \int x \, d\left(\frac{e^{-i\xi x}}{-i\xi}\right) = \frac{x e^{-i\xi x}}{-i\xi}\Bigg|_{0}^{1} + \int_{0}^{1} \frac{e^{-i\xi x}}{i\xi}\,dx$$

$$= -\frac{e^{-i\xi}}{i\xi} - \frac{1}{(i\xi)^2}\left[e^{-i\xi x}\right]_{0}^{1}$$

$$= -\frac{e^{-i\xi}}{i\xi} + \frac{1}{\xi^2}\left[e^{-i\xi} - 1\right]$$

Hence

$$\hat{g}(\xi) = \cancel{\frac{2\sin\xi}{\xi}} \cancel{- \frac{e^{i\xi} - e^{-i\xi}}{i\xi}} + \frac{2 - e^{i\xi} - e^{-i\xi}}{\xi^2}$$

$$= \boxed{\frac{2}{\xi^2}\left(1 - \cos\xi\right) \xrightarrow{|\xi| \to +\infty} 0 \text{ like } O\!\left(\frac{1}{|\xi|^2}\right)}$$

(c) $\hat{h}(\xi) = \int_{-1}^{1} (x^2-1)^2 e^{-i\xi x}\,dx$

$$= \int_{-1}^{1} \underbrace{(x^2-1)^2}_{\text{even}} \left(\cos(\xi x) + i\,\cancel{\sin(\xi x)}_{\text{odd}}\right) dx$$

$$= 2\int_0^1 (x^4 - 2x^2 + 1)\cos\xi x\,dx$$

$$\int x^4 \cos\xi x\,dx = \int x^4\,d\frac{\sin\xi x}{\xi}$$

$$= \frac{x^4\sin\xi x}{\xi} - 4\int x^3\frac{\sin\xi x}{\xi}\,dx$$

$$= \frac{x^4\sin\xi x}{\xi} + \frac{4}{\xi^2}\int x^3\,d(\cos\xi x)$$

$$= \frac{x^4\sin\xi x}{\xi} + \frac{4x^3\cos\xi x}{\xi^2} - \frac{12}{\xi^2}\int x^2\cos\xi x\,dx$$

$$\int x^2\cos\xi x\,dx = \int x^2\,d\frac{\sin\xi x}{\xi}$$

$$= \frac{x^2 \sin \xi x}{\xi} - 2\int \frac{x}{\xi} \sin \xi x \, dx$$

$$= \frac{x^2 \sin \xi x}{\xi} + \frac{2}{\xi^2} \int x \, d(\cos \xi x)$$

$$= \frac{x^2 \sin \xi x}{\xi} + \frac{2x \cos \xi x}{\xi^2} - \frac{2}{\xi^2} \int \cos \xi x \, dx$$

Hence

$$\int (x^4 - 2x^2 + 1) \cos \xi x \, dx$$

$$= \frac{x^4 \sin \xi x}{\xi} + \frac{4x^3 \cos \xi x}{\xi^2} - \frac{12}{\xi^2} \int x^2 \cos \xi x \, dx$$

$$- 2\left[\int x^2 \cos \xi x \, dx\right] + \int \cos \xi x \, dx$$

$$= \frac{x^4 \sin \xi x}{\xi} + \frac{4x^3 \cos \xi x}{\xi^2} + \frac{\sin \xi x}{\xi}$$

$$- \left(\frac{12}{\xi^2} + 2\right) \int x^2 \cos \xi x \, dx$$

$$= \frac{x^4 \sin \xi x}{\xi} + \frac{4x^3 \cos \xi x}{\xi^2} + \frac{\sin \xi x}{\xi}$$

$$- \left(\frac{12}{\xi^2} + 2\right)\left[\frac{x^2 \sin \xi x}{\xi} + \frac{2x \cos \xi x}{\xi^2} - \frac{2 \sin \xi x}{\xi^3}\right]$$

Hence

$$\int_0^1 (x^4 - 2x^2 + 1) \cos \xi x \, dx$$

$$= \cancel{\frac{\sin \xi}{\xi}} + \cancel{\frac{4 \cos \xi}{\xi^2}} + \cancel{\frac{\sin \xi}{\xi}}$$

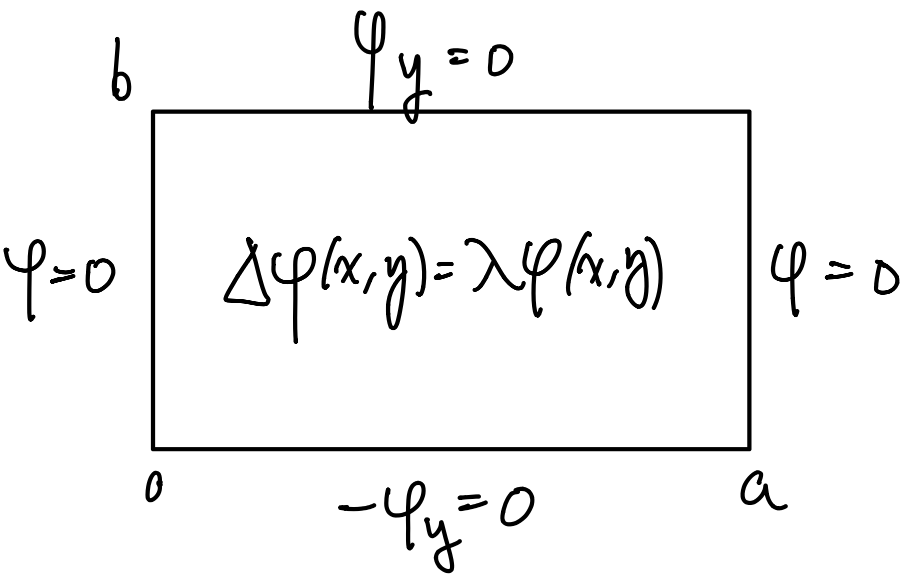
$$- \left(\frac{12}{\xi^2} + 2\right)\left[\cancel{\frac{\sin \xi}{\xi}} + \cancel{\frac{2 \cos \xi}{\xi^2}} - \frac{2 \sin \xi}{\xi^3}\right]$$

$$= -\frac{12}{\xi^3} \sin \xi - \frac{24 \cos \xi}{\xi^4} + \left(\frac{24}{\xi^2} + 4\right)\frac{\sin \xi}{\xi^3}$$

$$\boxed{\begin{array}{c} \hat{f}(\xi) = -\dfrac{16 \sin \xi}{\xi^3} - \dfrac{48 \cos \xi}{\xi^4} + \dfrac{48 \sin \xi}{\xi^5} \\ \xrightarrow{|\xi| \to +\infty} 0 \text{ like } O\left(\dfrac{1}{|\xi|^3}\right) \end{array}}$$

#3

(a)



M1 : Separation of variables:

$$\varphi(x,y) = X(x)Y(y)$$

$$\Delta\varphi = X''(x)Y(y) + X(x)Y''(y) = \lambda X(x)Y(y)$$

$$\frac{X''}{X} + \frac{Y''}{Y} = \lambda$$

$$\frac{X''(x)}{X(x)} = \lambda - \frac{Y''(y)}{Y(y)} = \mu$$

$$\frac{X''(x)}{X(x)} = \mu, \quad X(0) = X(a) = 0$$

$$\Longrightarrow \quad \boxed{X_n(x) = \sin\left(\frac{n\pi x}{a}\right), \quad \mu = -\frac{n^2\pi^2}{a^2} \quad n = 1, 2, 3, \cdots}$$

$$\frac{Y''(y)}{Y(y)} = \lambda - \mu, \quad Y'(0) = Y'(b) = 0$$

$$\Rightarrow \boxed{Y(y) = \cos\left(\frac{m\pi y}{b}\right), \quad \lambda - \mu = -\frac{m^2\pi^2}{b^2} \qquad m = \underline{0}, 1, 2, 3, \cdots}$$

$$\boxed{\begin{array}{l} \varphi_{nm}(x,y) = \sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right) \\ \lambda = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}, \qquad \begin{array}{l} n = \underline{1}, 2, 3, \cdots \\ m = \underline{0}, 1, 2, 3, \cdots \end{array} \end{array}}$$

(Note: $\lambda \neq 0$ for any m, n.)

M2: Use [F. p.121, Thm 4.1] directly:

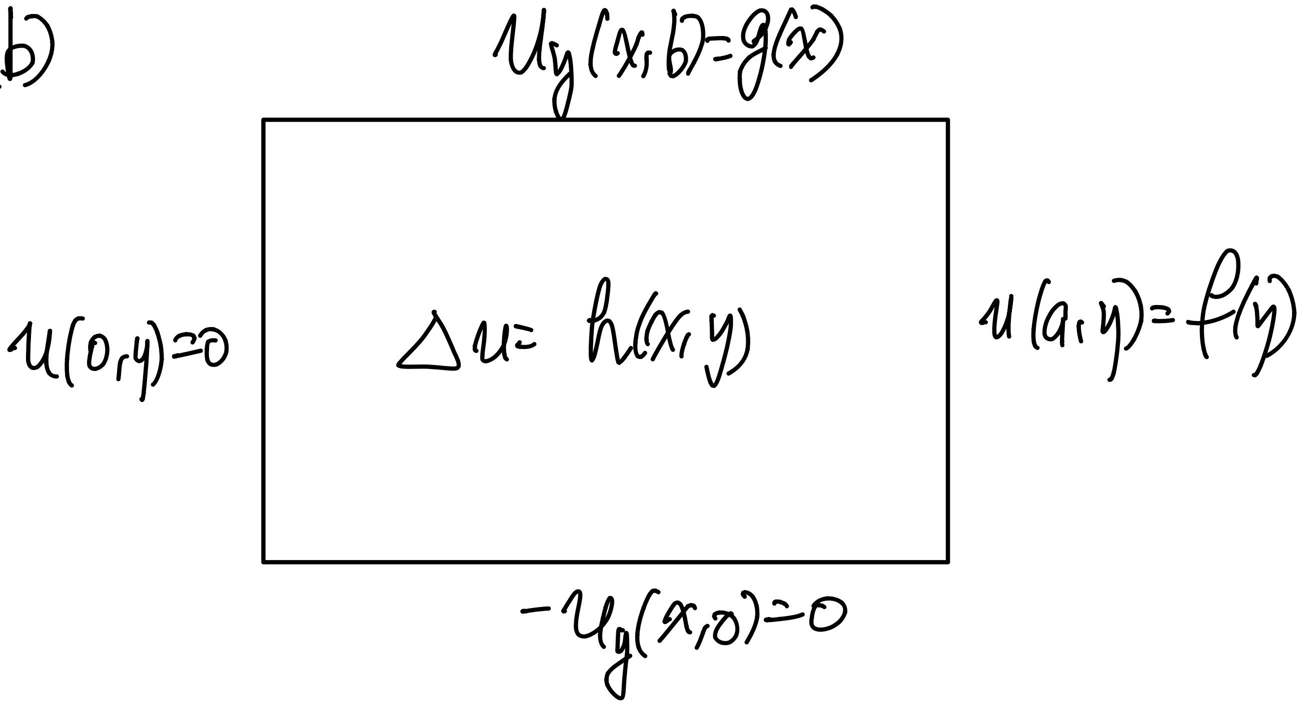
$$\varphi(x,y) = \sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)$$

Dir. B.C. in x-direction

Neu. B.C. in y-direction

$$\lambda = -\frac{n^2\pi^2}{a^2} - \frac{m^2\pi^2}{b^2}, \qquad \begin{array}{l} n = 1, 2, 3, \cdots \\ m = 0, 1, 2, \cdots \end{array}$$

(b)



$$u = q + v_1 + v_2$$

(i) $\Delta q = h$, with homog. B.C.

Let $q(x,y) = \sum_{\substack{n=1 \\ m=0}}^{\infty} C_{nm} \sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)$

$$h(x,y) = \sum_{\substack{n=1 \\ m=0}}^{\infty} d_{nm} \sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)$$

$$\Updownarrow$$

$$\Delta q = \sum_{\substack{n=1 \\ m=0}}^{\infty} C_{nm}\lambda_{nm} \sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)$$

$$\Longrightarrow \boxed{C_{nm} = \frac{d_{nm}}{\lambda_{nm}}}$$

$\lambda_{nm} = -\left(\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}\right)$

(note, $\lambda_{nm} \neq 0$)

$$\boxed{d_{nm} = \frac{\left\langle h,\ \sin\frac{n\pi x}{a}\cos\frac{m\pi y}{b}\right\rangle}{\left\| \sin\frac{n\pi x}{a}\cos\frac{m\pi y}{b}\right\|^2}}$$

$$\int_0^b\int_0^a h(x,y)\sin\frac{n\pi x}{a}\cos\left(\frac{m\pi y}{b}\right)dx\,dy$$

$$= \int_0^b\int_0^a \sin^2\frac{n\pi x}{a}\cos^2\frac{m\pi y}{b}\,dx\,dy$$

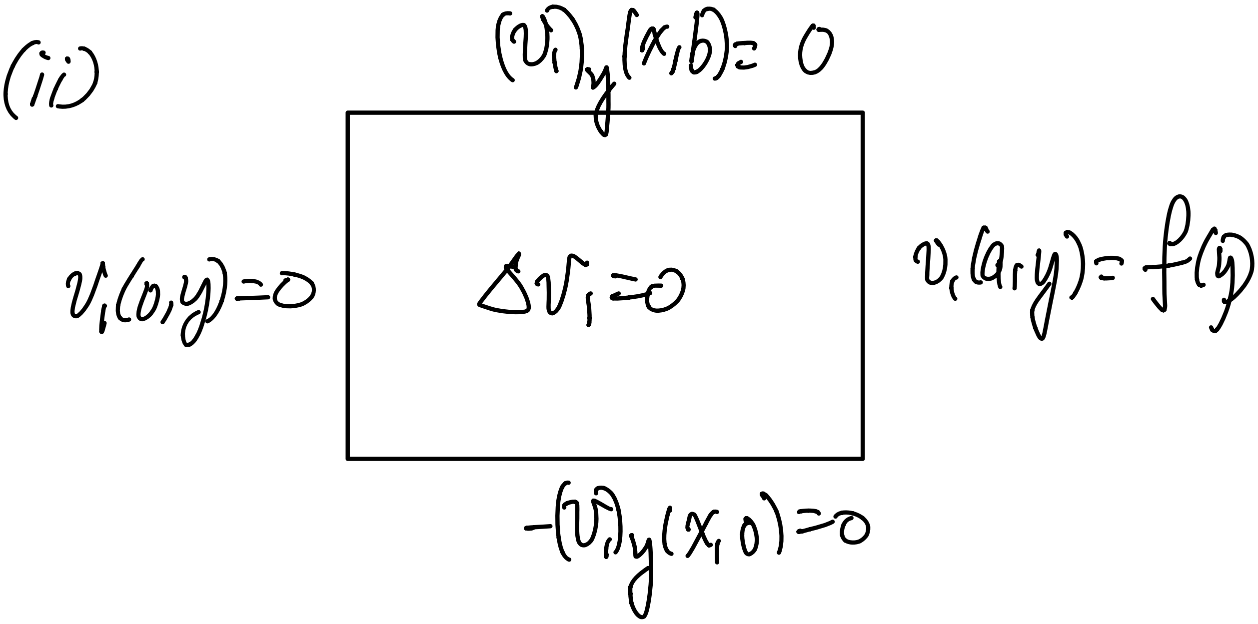
$$= \left(\int_0^a \sin^2\frac{n\pi x}{a}\,dx\right)\left(\int_0^b \cos^2\frac{m\pi y}{b}\,dy\right)$$

$$\frac{a}{2}\ (n\neq 0) \qquad \begin{cases} b & \text{if } m=0 \\ \frac{b}{2} & \text{if } m\neq 0 \end{cases}$$

$$\boxed{g(x,y) = \sum_{\substack{n=1\\m=0}}^{\infty} \frac{d_{nm}}{\lambda_{nm}}\sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)}$$

$$-\left(\frac{n^2\pi^2}{a^2}+\frac{m^2\pi^2}{b^2}\right) \quad (\text{Note} \neq 0)$$

(ii)



Use separation of variable:

$$v_1(x,y) = X(x)Y(y)$$

$$\Rightarrow \begin{cases} Y(y) = \cos\left(\frac{m\pi y}{b}\right), & m = 0,1,2,\cdots \\ X(x) = \begin{cases} A_m e^{\frac{m\pi}{b}x} + B_m e^{-\frac{m\pi}{b}x} & m=1,2,\cdots \\ A_0 + B_0 x & m = 0 \end{cases} \end{cases}$$

$m=0$

$$v_1(x,y) = (A_0 + B_0 x) + \sum_{m=1}^{\infty} \left(A_m e^{\frac{m\pi x}{b}} + B_m e^{-\frac{m\pi x}{b}}\right) \cos\frac{m\pi y}{b}$$

$\boxed{x=0} \Longrightarrow v_i(0,y)=0$

$$0 = A_0 + \sum_{m=1}^{\infty} (A_m + B_m) \cos\left(\frac{m\pi y}{b}\right)$$

$$\Longrightarrow A_0 = 0, \quad A_m + B_m = 0, \text{ i.e. } B_m = -A_m$$

$$\Longrightarrow v_i(x,y)$$

$$= B_0 x + \sum_{m=1}^{\infty} A_m \left(e^{\frac{m\pi x}{b}} - e^{-\frac{m\pi x}{b}}\right) \cos\left(\frac{m\pi y}{b}\right)$$

$x=a \Longrightarrow v_i(a,y) = f(y)$

$$f(y) = \underline{B_0 a} + \sum_{m=1}^{\infty} \underline{A_m \left(e^{\frac{m\pi a}{b}} - e^{-\frac{m\pi a}{b}}\right)} \cos\left(\frac{m\pi y}{b}\right)$$

$m=0$

$m\neq 0$

$$B_0 a = \frac{\int_0^b f(y) \cdot 1 \, dy}{\int_0^b 1^2 \, dy} = \frac{1}{b} \int_0^b f(y)\, dy$$

$$\boxed{B_0 = \frac{1}{ab} \int_0^b f(y)\, dy}$$

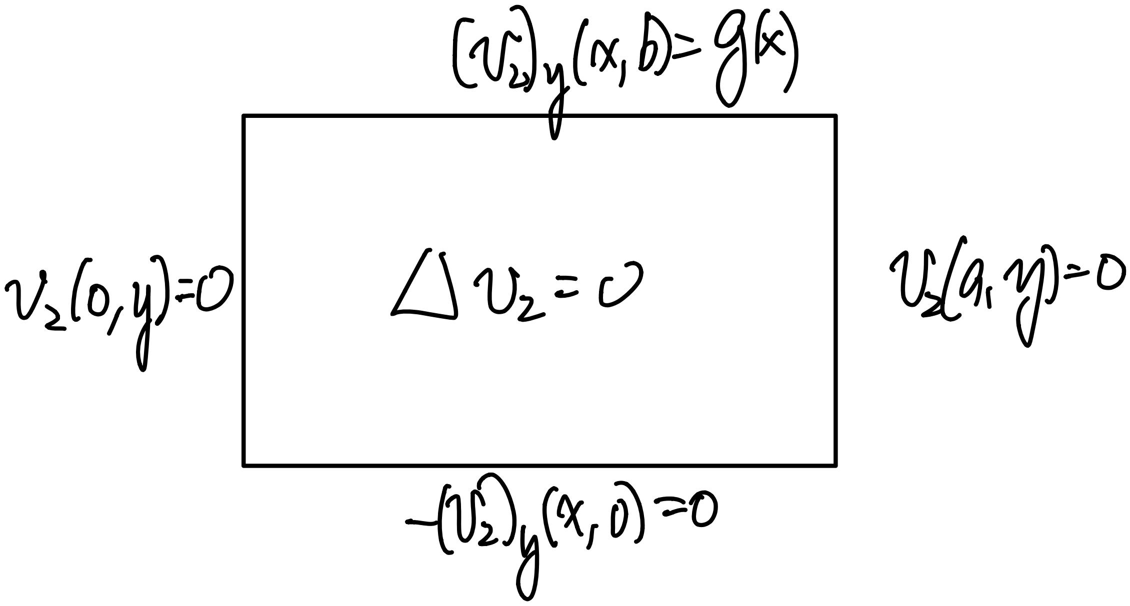
$$A_m\left(e^{\frac{m\pi a}{b}} - e^{-\frac{m\pi a}{b}}\right) = \frac{\int_0^b f(y)\cos\left(\frac{m\pi y}{b}\right)dy}{\int_0^b \cos^2\left(\frac{m\pi y}{b}\right)dy}$$

$$= \frac{2}{b}\int_0^b f(y)\cos\frac{m\pi y}{b}\,dy$$

$$\boxed{A_m = \frac{2}{b}\,\frac{\int_0^b f(y)\cos\frac{m\pi y}{b}\,dy}{e^{\frac{m\pi a}{b}} - e^{-\frac{m\pi a}{b}}}}$$

$$\boxed{v_1(x,y) = B_0 x + \sum_{m=1}^{\infty} A_m\left(e^{\frac{m\pi x}{b}} - e^{-\frac{m\pi x}{a}}\right)\cos\left(\frac{m\pi y}{b}\right)}$$

(ii)



Use separation of variables again:

$$u_2(x,y) = X(x)Y(y)$$

$$X(x) = \sin\left(\frac{n\pi x}{a}\right), \quad n = 1, 2, 3, \cdots$$

$$Y(y) = \left(A_n e^{\frac{n\pi y}{a}} + B_n e^{-\frac{n\pi y}{a}}\right)$$

$$\Longrightarrow u_2(x,y) = \sum_{n=1}^{\infty} \left(A_n e^{\frac{n\pi y}{a}} + B_n e^{-\frac{n\pi y}{a}}\right) \sin\left(\frac{n\pi x}{a}\right)$$

$$(u_2)_y(x,y) = \sum_{n=1}^{\infty} \left(\frac{n\pi}{a}\right)\left[A_n e^{\frac{n\pi y}{a}} - B_n e^{-\frac{n\pi y}{a}}\right] \sin\left(\frac{n\pi x}{a}\right)$$

$$y=0 \Longrightarrow (u_2)_y(x,0) = 0$$

$$\Longrightarrow A_n = B_n$$

$$y=0 \Longrightarrow (u_2)_y(x,b) = g(x)$$

$$\Longrightarrow g(x) = \sum_{n=1}^{\infty} \underbrace{\left(\frac{n\pi}{a}\right) A_n \left[e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}}\right]} \sin\left(\frac{n\pi x}{a}\right)$$

$$\left(\frac{n\pi}{a}\right) A_n\left[e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}}\right] = \frac{\int_0^a g(x)\sin\frac{n\pi x}{a}\,dx}{\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)dx}$$

$$A_n = \left(\frac{a}{n\pi}\right)\frac{\int_0^a g(x)\sin\frac{n\pi x}{a}\,dx}{\frac{a}{2}\left(e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}}\right)}$$

$$= \boxed{\frac{2}{n\pi}\,\frac{\int_0^a g(x)\sin\left(\frac{n\pi x}{a}\right)dx}{\left(e^{\frac{n\pi b}{a}} - e^{-\frac{n\pi b}{a}}\right)}}$$

$$\boxed{u_2(x) = \sum_{n=1}^{\infty} A_n\left(e^{\frac{n\pi y}{a}} + e^{-\frac{n\pi y}{a}}\right)\sin\left(\frac{n\pi x}{a}\right)}$$

Final solution:

$$\boxed{\begin{aligned} u &= q + v_1 + v_2 \\ &= \sum_{\substack{n=1 \\ m=0}}^{\infty} \frac{-d_{nm}}{\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}} \sin\left(\frac{n\pi x}{a}\right) \cos\left(\frac{m\pi y}{b}\right) \\ &\quad + B_0 x + \sum_{m=1}^{\infty} A_n \left(e^{\frac{m\pi x}{b}} - e^{-\frac{m\pi x}{b}} \right) \cos\frac{m\pi y}{b} \\ &\quad + \sum_{n=1}^{\infty} A_n \left(e^{\frac{n\pi y}{a}} + e^{-\frac{n\pi y}{a}} \right) \sin\left(\frac{n\pi x}{a}\right) \end{aligned}}$$

(no compatibility condition needed.)

#4 (a) $u_t = D u_{xx} + a u_x + bu$

$\mathcal{F}$

$$\hat{u}_t = -D\xi^2 \hat{u} + ai\xi\hat{u} + b\hat{u}$$

$$\hat{u}_t = (-D\xi^2 + ai\xi + b)\hat{u}$$

$$\hat{u}(\xi,t) = e^{(-D\xi^2 + ai\xi + b)t}\hat{u}(\xi,0)$$

$\mathcal{F}^{-1}$

$$= e^{bt} e^{ai\xi t} \underbrace{e^{-D\xi^2 t} \hat{f}(\xi)}_{(f*g)(x,t)}$$

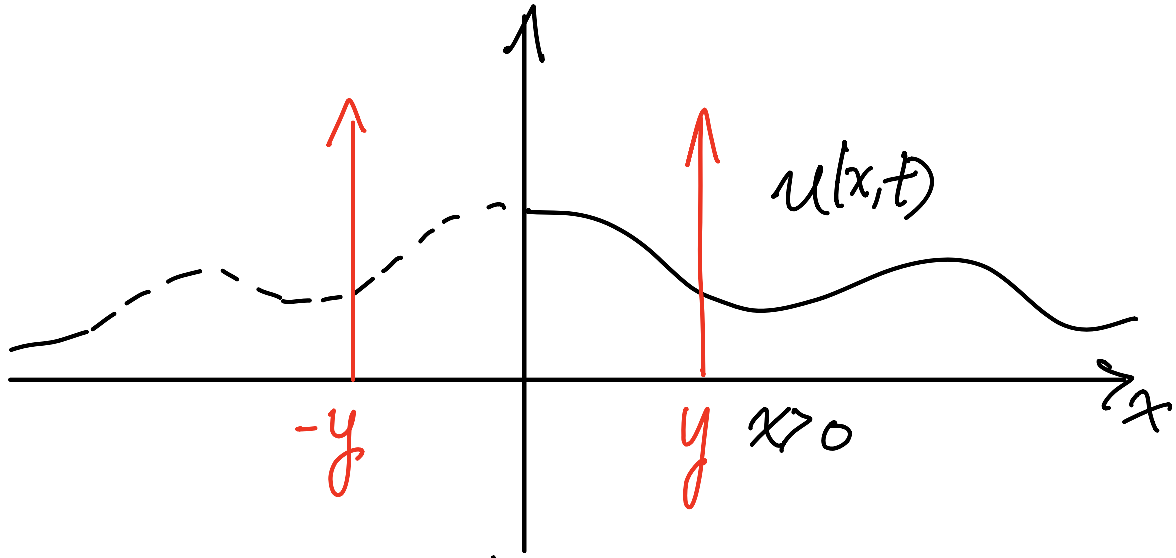
shift in x by at

$$g(x,t) = \frac{e^{-\frac{x^2}{4Dt}}}{\sqrt{4D\pi t}}$$

$$(f*g)(x+at,t)$$

$$\boxed{u(x,t) = e^{bt}\int_{-\infty}^{\infty} \frac{e^{-\frac{(x+at-y)^2}{4Dt}}}{\sqrt{4D\pi t}} f(y)\,dy}$$

(b)



Consider even extension:

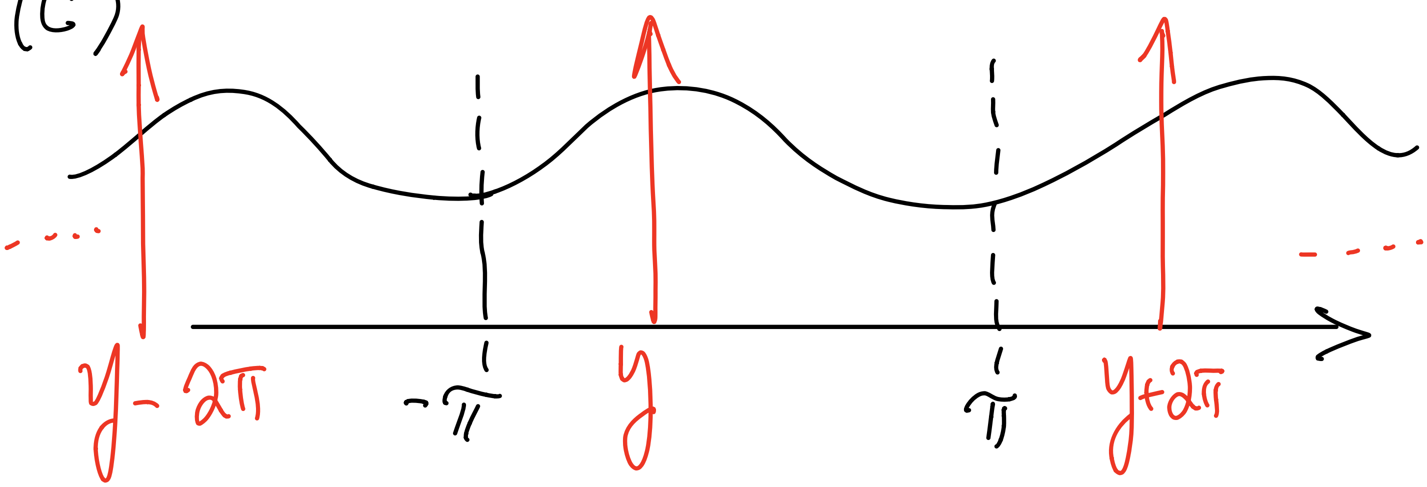
$$\tilde{u}(x,t) = \begin{cases} u(x,t) & x>0 \\ u(-x,t), & x<0 \end{cases}$$

$$\tilde{f}(x) = \begin{cases} f(x) & x>0 \\ f(-x) & x<0 \end{cases}$$

$$\tilde{u}(x,t) = \int_{-\infty}^{\infty} \tilde{f}(y)\, g(x-y,t)\, dy$$

$$\boxed{u(x,t) = \int_0^{\infty} f(y) \underbrace{\left[g(x-y,t) + g(x+y,t)\right]}_{\text{Heat Kernel}} dy}$$

(C)



$$\boxed{u(x,t)=\int_{-\pi}^{\pi} f(y)\underbrace{\left[\sum_{n=-\infty}^{\infty} g(x-y-2n\pi,t)\right]}_{\text{Heat kernel}} dy}$$

#5 $g(x)=\sum_{k=-\infty}^{\infty} f(x+2k\pi)$

(a) $g(x+2\pi)=\sum_{k=-\infty}^{\infty} f(x+2\pi+2k\pi)$

$$=\sum_{k=-\infty}^{\infty} f(x+\underbrace{(k+1)}_{\tilde{k}}2\pi)$$

$$=\sum_{\tilde{k}=-\infty}^{\infty} f(x+\tilde{k}2\pi) = g(x)$$

(b) $g(x)=\sum_{n=-\infty}^{\infty} c_n e^{inx}$

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-inx}\,dx$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} f(x+2\pi k)e^{-inx}\,dx$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} f(\underbrace{x+2\pi k}_{y}) e^{-inx} \, dx$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{2\pi k - \pi}^{2\pi k + \pi} f(y) e^{-in(y-2\pi k)} \, dy$$

$$(e^{in2\pi k} = 1)$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{\pi(2k-1)}^{\pi(2k+1)} f(y) e^{-iny} \, dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) e^{-iny} \, dy$$

$$\boxed{c_n = \frac{1}{2\pi} \hat{f}(n)}$$

(c) $$g(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$\sum_{k=-\infty}^{\infty} f(x+2k\pi) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \hat{f}(n) e^{inx}$$

$$\Big\downarrow \; x=0$$

$$\sum_{k=-\infty}^{\infty} f(2k\pi) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \hat{f}(n)$$

$$\Updownarrow$$

$$\boxed{\sum_{n=-\infty}^{\infty} \hat{f}(n) = 2\pi \sum_{n=-\infty}^{\infty} f(2n\pi)}$$

(d) $$e^{-|x|} \xrightarrow{\mathcal{F}} \frac{2}{\xi^2+1}$$

[F, p. 223, #11]

From (c):

$$\sum_{n=-\infty}^{\infty} \frac{2}{n^2+1} = 2\pi \sum_{n=-\infty}^{\infty} e^{-2\pi|n|}$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+1} = \pi\left[1 + 2\underbrace{\sum_{n=1}^{\infty} e^{-2\pi n}}\right]$$

$$\left(r + r^2 + r^3 + \cdots = \frac{r}{1-r} \quad \text{for } |r|<1\right)$$

$$= \pi\left[1 + 2e^{-2\pi}\frac{1}{1-e^{-2\pi}}\right]$$

$$= \pi\left[\frac{1+e^{-2\pi}}{1-e^{-2\pi}}\right]$$

$$= \pi\left[\frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}\right] = \boxed{\pi \coth(\pi)}$$