Examples of Differential Equations

(1) Population Dynamics (Single Species:)

$x(t)$ = population at time t.

(a) Exponential Growth:

$$\boxed{\frac{dx}{dt} = rx}$$

$r = \frac{1}{x}\frac{dx}{dt}$ = growth rate per capita

(b) Logistic Growth:

$$\boxed{\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right)}$$

K = Carrying Capacity.

Examples of Differential Equations

(1) Population Dynamics (Interacting Species)

(2) Competing species:

$x = x(t)$, $y = y(t)$:

population of 2 species consuming the same food / resources.

$$\boxed{\begin{aligned} \frac{dx}{dt} &= r_1 x(1 - Ax - By) \\ \frac{dy}{dt} &= r_2 y(1 - Cx - Dy) \end{aligned}}$$

Examples of Differential Equations

(1) Population Dynamics (Interacting Species)

(b) Predator-Prey

$x = x(t)$: population of prey

$y = y(t)$: population of predator

$$\boxed{\begin{aligned} \frac{dx}{dt} &= r_1 x(1 - Ax - By) \\ \frac{dy}{dt} &= r_2 y(Cx - Dy) \end{aligned}}$$

Examples of Differential Equations

(1) Population Dynamics (Interacting Species.)

(c) Epidemic Model (SIR)

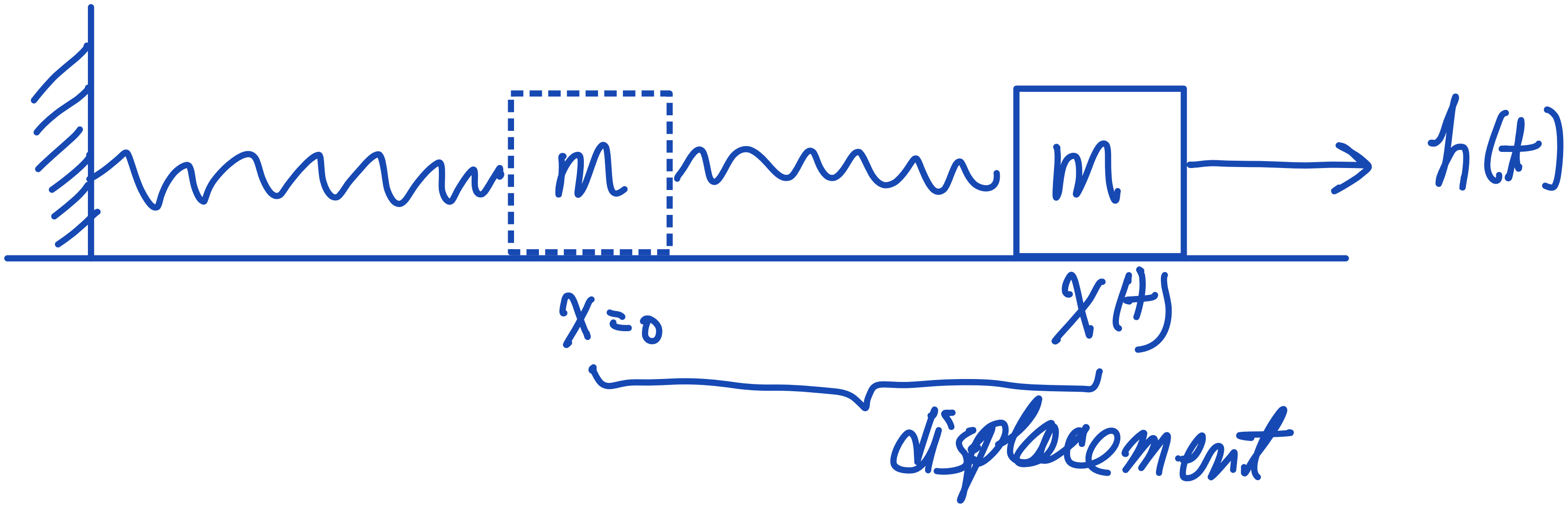
$$S \xrightarrow{\alpha} I \xrightarrow{\beta} R$$

susceptible infected recovered/dead

$$\boxed{\begin{aligned} \frac{dS}{dt} &= \alpha S I \\ \frac{dI}{dt} &= \alpha S I - \beta I \\ \frac{dR}{dt} &= \beta I \end{aligned}}$$

Examples of Differential Equations

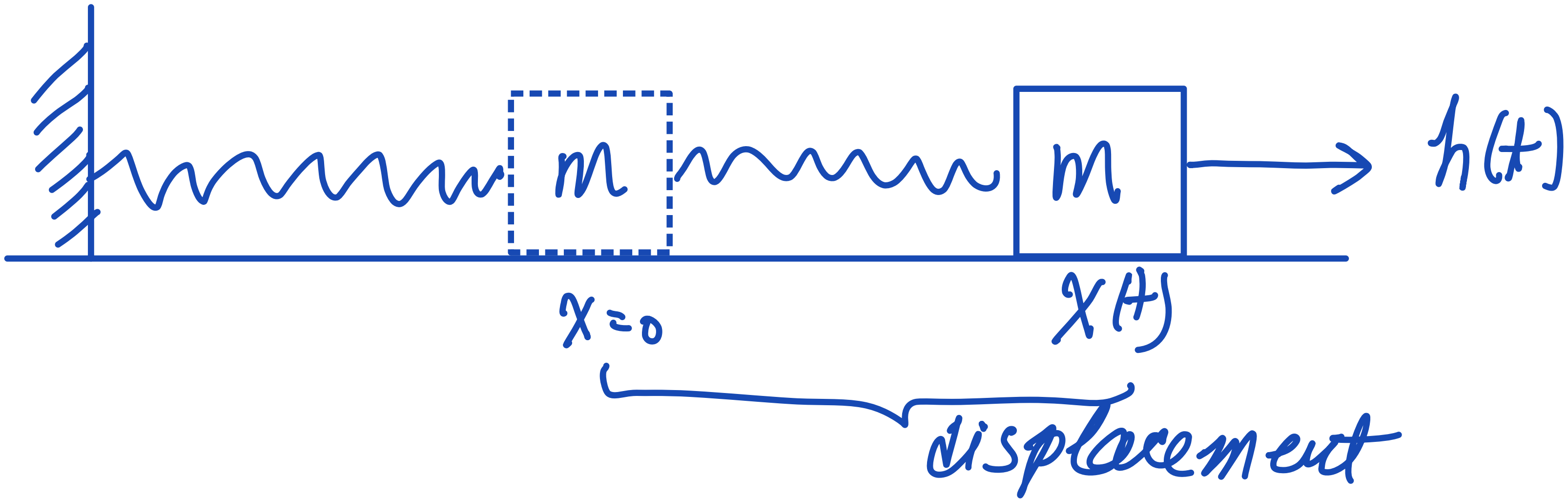
(1) (Linear) Harmonic Oscillator



$$\boxed{m\ddot{x} = -kx - \gamma\dot{x} + h(t)}$$

Examples of Differential Equations

(1) (Linear) Harmonic Oscillator



$$\boxed{m\ddot{x} = -kx - \gamma\dot{x} + h(t)}$$

ma

Spring force

friction

$\gamma > 0$

external force

(2) Nonlinear Spring

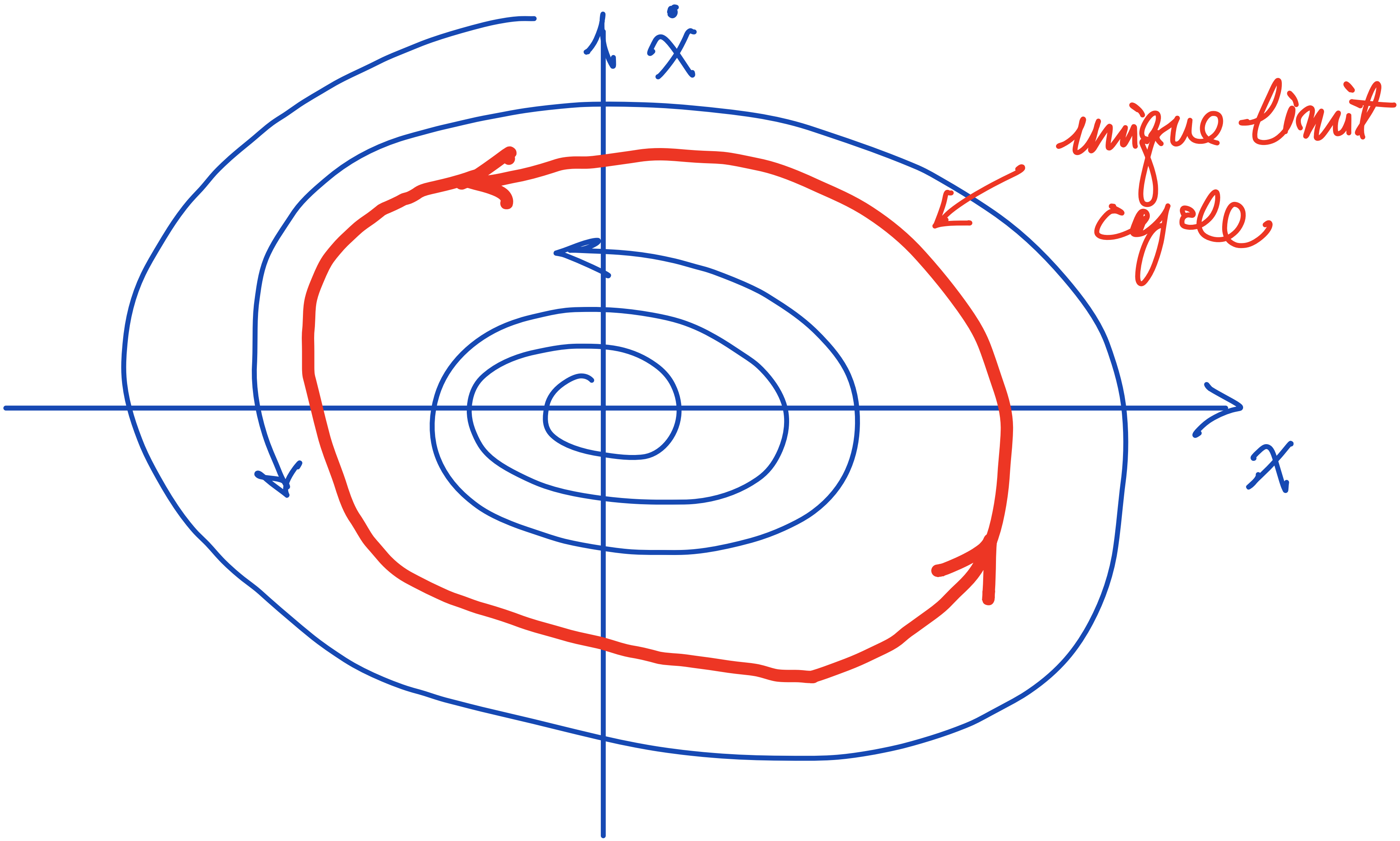
van der Pol oscillator : nonlinear friction

$$\boxed{m\ddot{x} = -kx - \alpha\,\phi(x)\,\dot{x} + h(t)}$$

Frictional coefficient $\quad \phi(x) = (x^2 - 1) = \begin{cases} < 0 & \text{if } |x| < 1 \\ > 0 & \text{if } |x| > 1 \end{cases}$

(2) Nonlinear Spring

Van der Pol oscillator : nonlinear friction



(3) Nonlinear Spring

Duffing's oscillator: nonlinear spring

$$\boxed{m\ddot{x} = (x - x^3) - \gamma\dot{x} + h(t)}$$

(3) Nonlinear Spring

Duffing's oscillator: nonlinear spring

$$\boxed{m\ddot{x} = (x - x^3) - \gamma\dot{x} + h(t)}$$

$$\boxed{x - x^3 = -\frac{\partial}{\partial x}V(x)\,, \text{ where } V(x) = \frac{(x^2-1)^2}{4}}$$

potential function

(3) Nonlinear Spring

Duffing's oscillator: nonlinear spring

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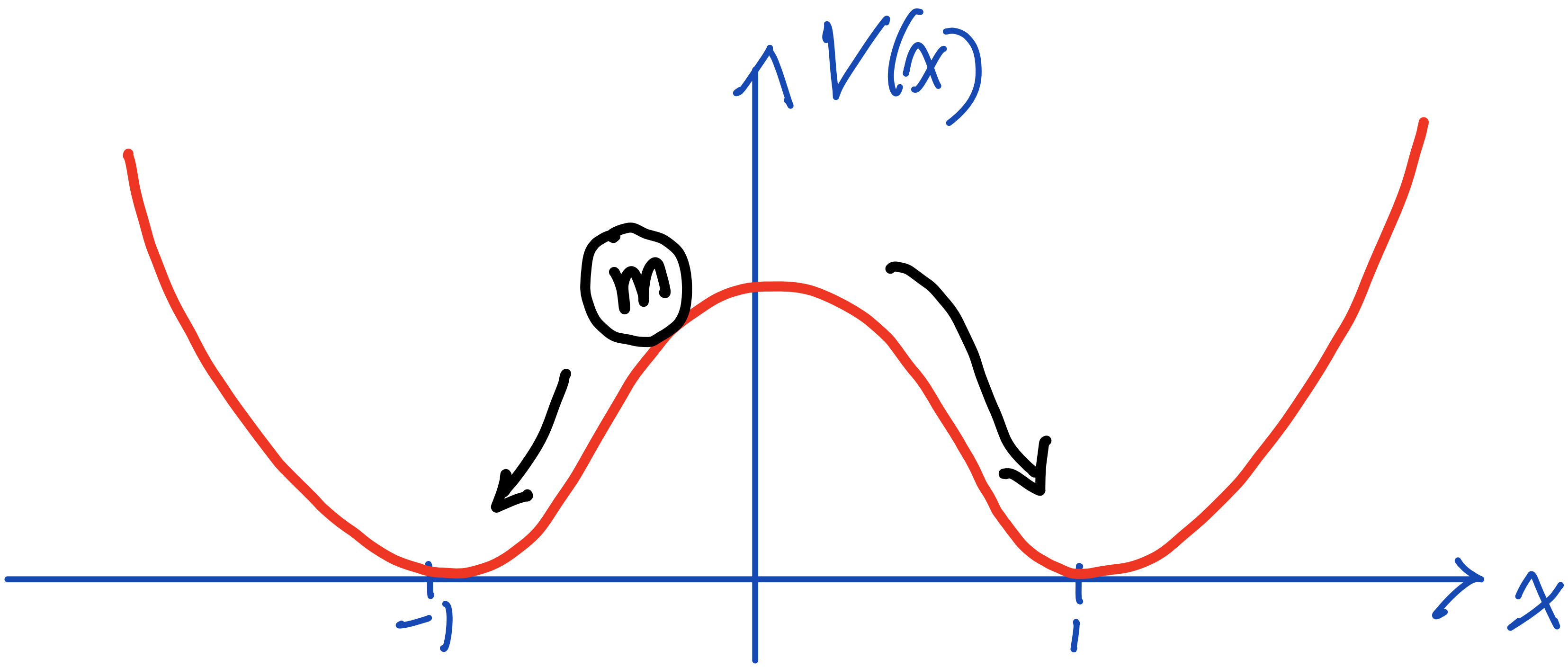

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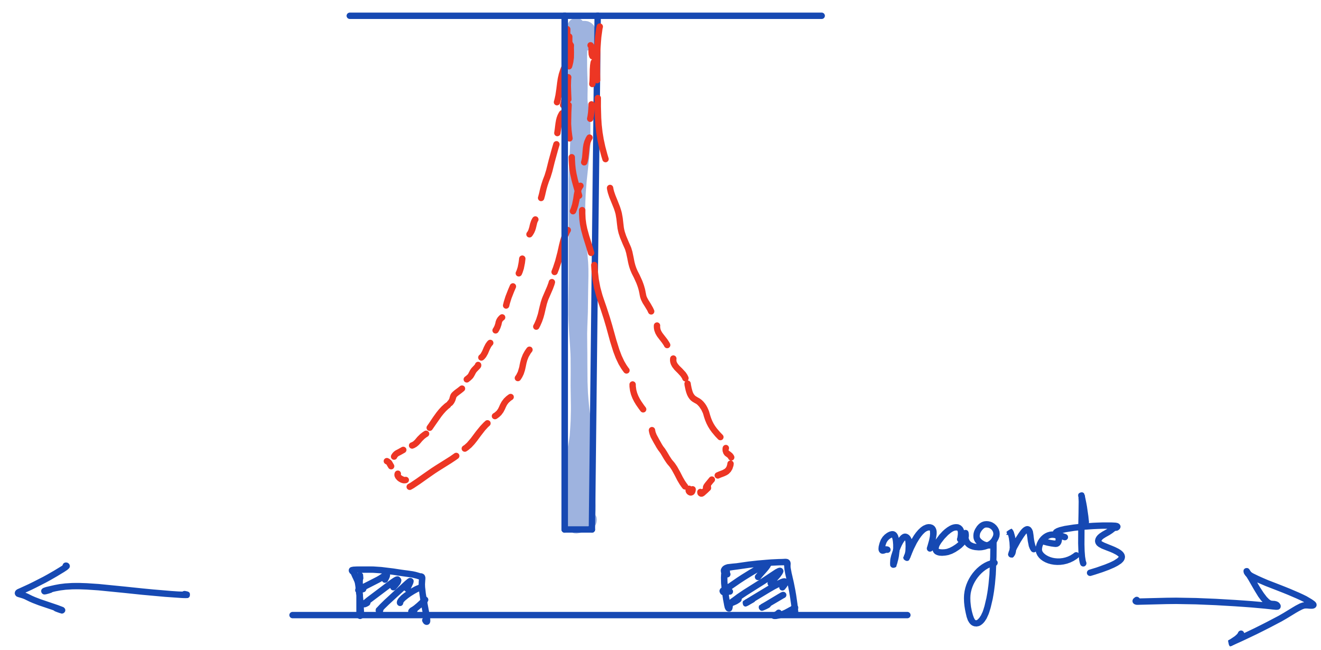

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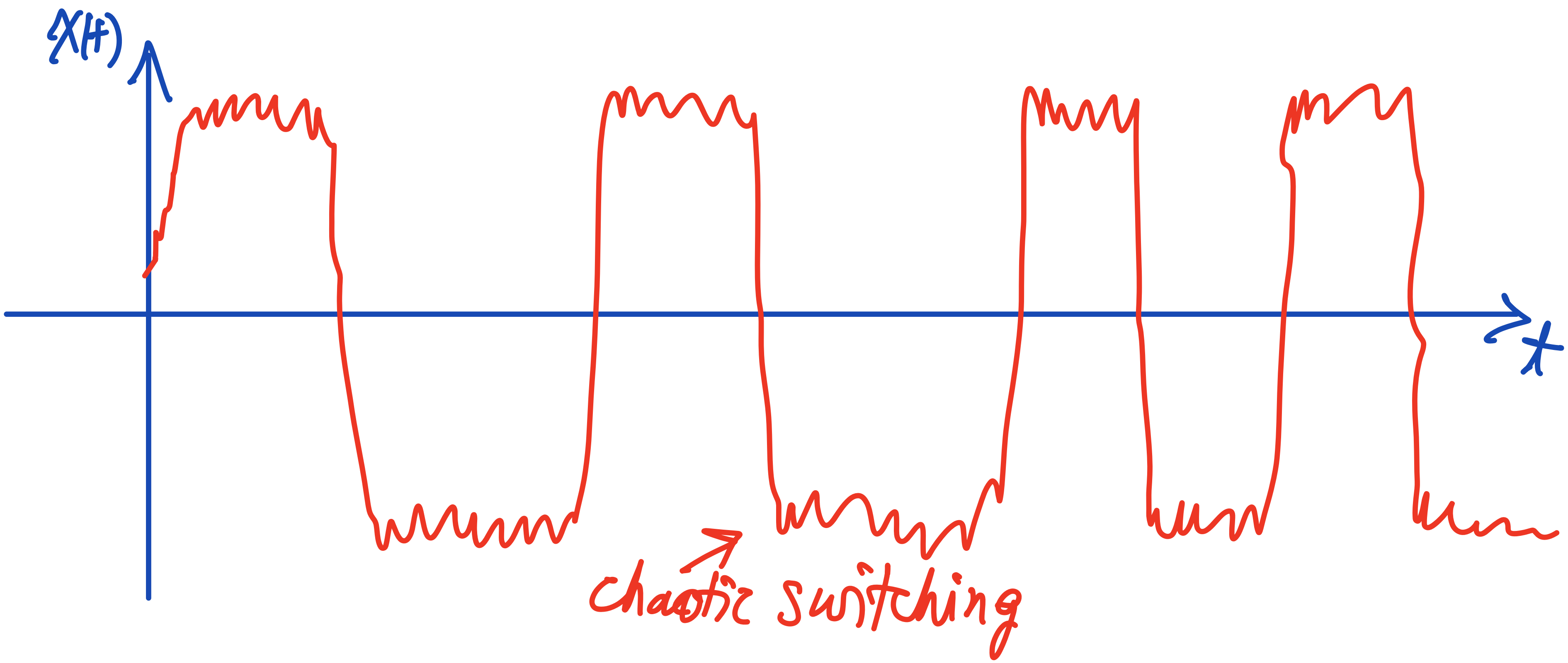
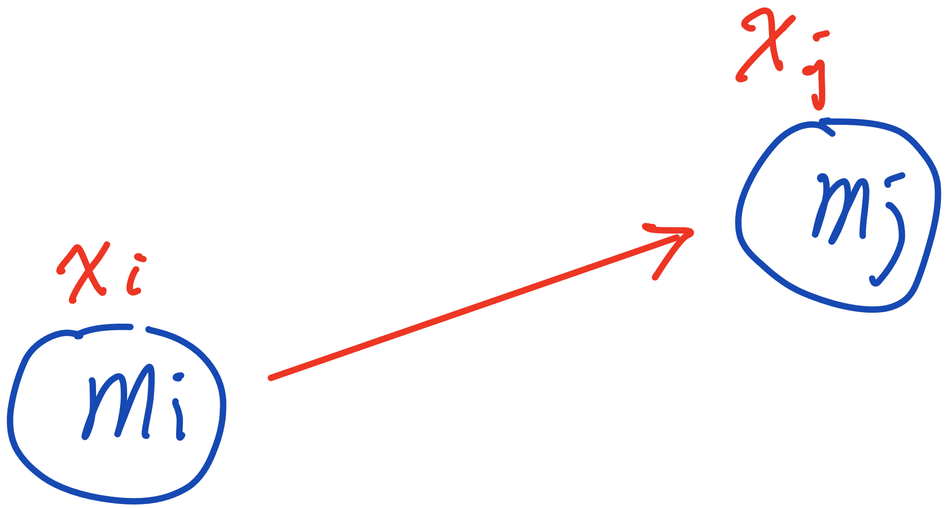


(4) Celestial Mechanics

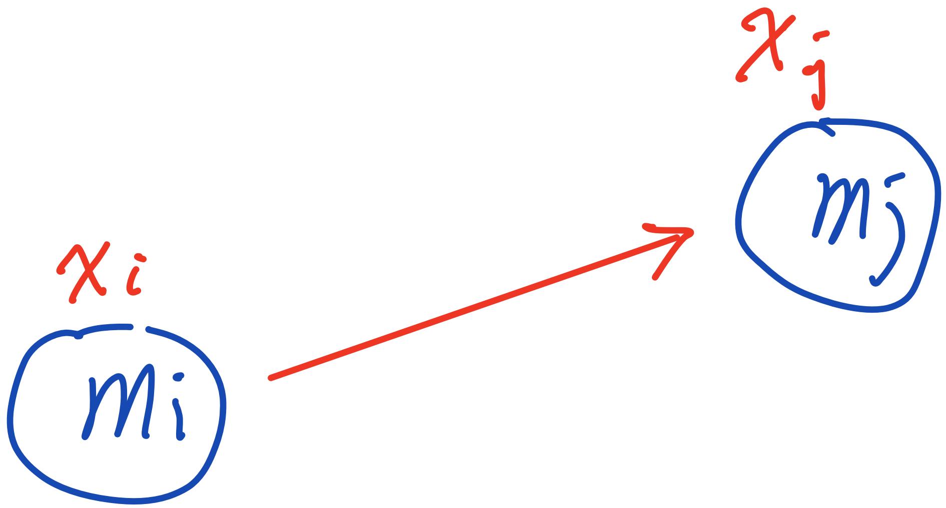
(Newton's Gravitational Motion)



$$\boxed{F_{ij} = G m_i m_j \frac{x_j - x_i}{|x_j - x_i|^3}}$$

(4) Celestial Mechanics

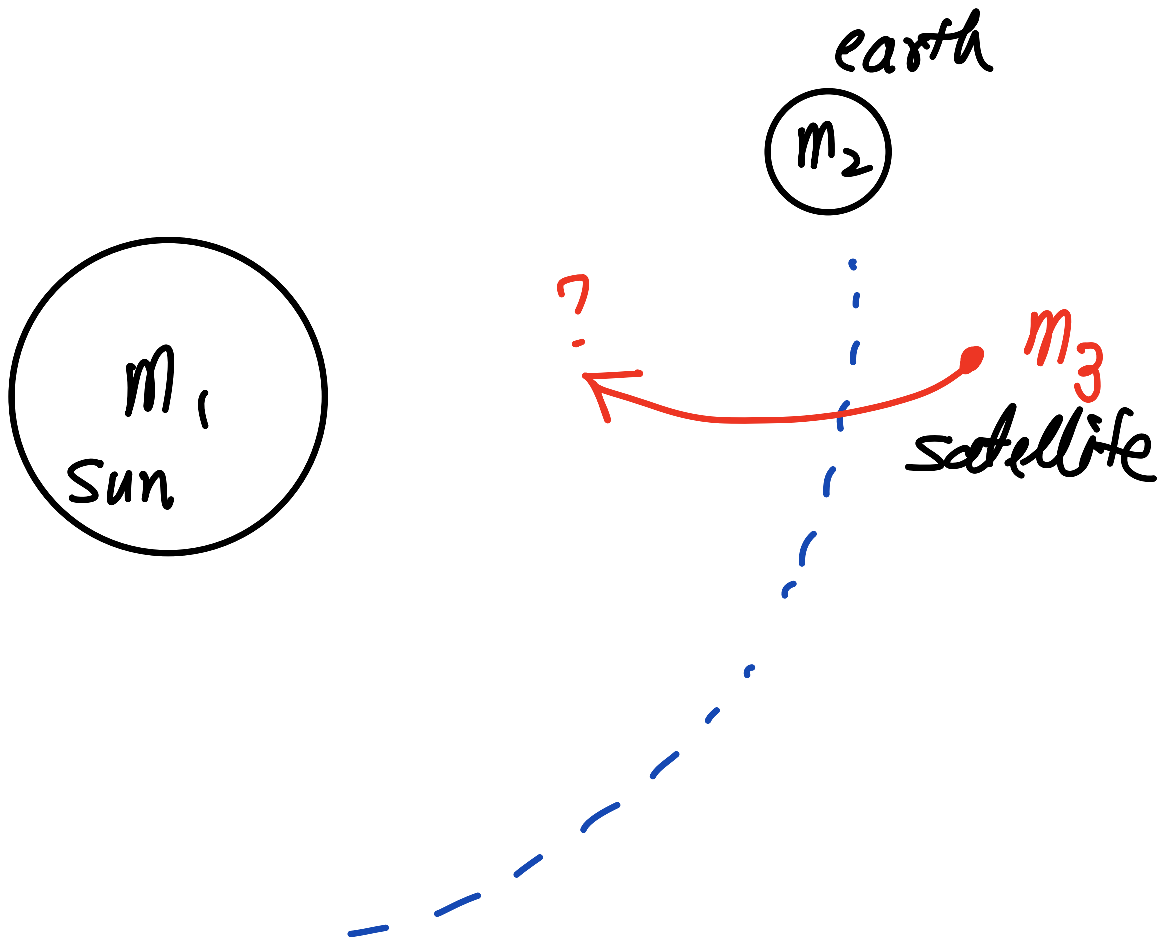
(Newton's Gravitational Motion)



$$\boxed{m_i \ddot{x}_i = \sum_{j \neq i} G m_i m_j \frac{x_j - x_i}{|x_j - x_i|^3}}$$

(4) PCR3BP (Planar Circular Restricted Three Body Problem)

m_1 (sun), m_2 (earth) $\gg$ m_3 (satellite)



(4) PCR3BP (Planar Circular Restricted Three Body Problem)

m_1 (sun), m_2 (earth) $\gg$ m_3 (satellite)

motions predetermined (known)

to be determined

$$\boxed{\ddot{x}_3 = (Gm_1)\frac{x_1 - x_3}{|x_1 - x_3|^3} + (Gm_2)\frac{x_2 - x_3}{|x_2 - x_3|^3}}$$

(5) Hamiltonian Dynamics

State variables: $X(t) \in \mathbb{R}^n$ (position)

$Y(t) \in \mathbb{R}^n$ (momentum)

Hamiltonian function: $H(X, Y) \in \mathbb{R}$

Dynamics:

$$\boxed{\begin{aligned} \dot{X} &= \nabla_Y H(X, Y) \\ \dot{Y} &= -\nabla_X H(X, Y) \end{aligned}}$$

(5) Hamiltonian Dynamics

State variables: $X(t) \in \mathbb{R}^n$ (position)

$Y(t) \in \mathbb{R}^n$ (momentum)

Hamiltonian function: $H(X,Y) \in \mathbb{R}$

Dynamics:

$$\boxed{\begin{aligned} \dot{X} &= \nabla_Y H(X,Y) \\ \dot{Y} &= -\nabla_X H(X,Y) - \gamma \dot{X} \end{aligned}}$$

(friction)

(5) Hamiltonian Dynamics

State variables: $X(t) \in \mathbb{R}^n$ (position)

$Y(t) \in \mathbb{R}^n$ (momentum)

Hamiltonian function:

$$\boxed{H(X,Y) = V(x) + K(Y)}$$

potential energy   kinetic energy

(5) Hamiltonian Dynamics

State variables: $X(t) \in \mathbb{R}^n$ (position)

$Y(t) \in \mathbb{R}^n$ (momentum)

Dynamics:

$$\dot{X} = \nabla_Y K(Y)$$

$$\dot{Y} = -\nabla_X V(X) \left(-\gamma \dot{X}\right)$$

(5) Hamiltonian Dynamics

e.g. Harmonic oscillator:

$$m\ddot{x} = -kx - \gamma\dot{x}$$

$$\boxed{\begin{aligned} H(x,y) &= \frac{1}{2}kx^2 + \frac{1}{2m}y^2 \\ \dot{x} &= H_y(x,y) = \frac{1}{m}y \\ \dot{y} &= -H_x(x,y) = -kx - \gamma\dot{x} \end{aligned}}$$

$(y = m\dot{x})$

$(m\ddot{x} = -kx - \gamma\dot{x})$

(6) Chaotic Behavior

(sensitive dependence on initial data:
$$|x_1(t) - x_2(t)| \sim e^{\lambda t}|x_1(0) - x_2(0)|)$$

Lorenz Model

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - xz - y$$
$$\dot{z} = xy - bz$$

(6) Chaotic Behavior

(sensitive dependence on initial data:

$$|x_1(t) - x_2(t)| \sim e^{\lambda t} |x_1(0) - x_2(0)|)$$

Lorenz Model (Lorenz attractor)

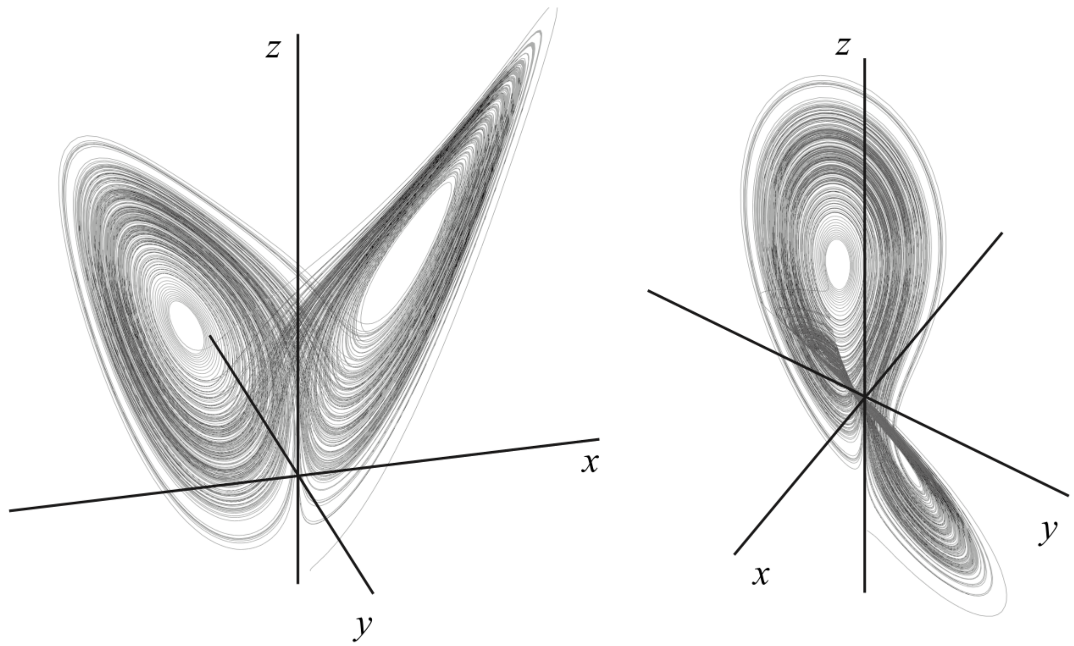


Figure 4.20. *Two views of a numerical approximation of the Lorenz Attractor for* $(\sigma, b, r) = (10, 8/3, 28)$. *The axes shown are centered at* $(0,0,20)$ *and are of length* 50.

(6) Chaotic Behavior

(sensitive dependence on initial data:

$$|x_1(t) - x_2(t)| \sim e^{\lambda t}|x_1(0) - x_2(0)|)$$

ABC Flow

$$\dot{x} = A\sin z + C\cos y$$

$$\dot{y} = B\sin x + A\cos z$$

$$\dot{z} = C\sin y + B\cos x$$

(6) Chaotic Behavior

(sensitive dependence on initial data:

$$|x_1(t) - x_2(t)| \sim e^{\lambda t} |x_1(0) - x_2(0)|\,)$$

ABC Flow (Arnold–Beltrami–Childress)

$$\left.\begin{aligned} \dot{x} &= A\sin z + C\cos y \\ \dot{y} &= B\sin x + A\cos z \\ \dot{z} &= C\sin y + B\cos x \end{aligned}\right\} \quad V(x,y,z): \left(\begin{aligned} &\operatorname{div} V = 0 \\ &v = \nabla \times v \end{aligned} \right)$$

(6) Chaotic Behavior (Simplest chaotic sys)

(M, p.21)

Table 1.1. *Quadratic, chaotic differential equations.*

Sprott's #	ODE	Reduced Parameters (others set to +1)	Chaotic Parameter Values
B	$\dot{x}=ayz,\quad \dot{y}=bx-cy$ $\dot{z}=d-exy\quad (ae>0)$	d	$d=1$
C	$\dot{x}=ayz,\quad \dot{y}=bx-cy$ $\dot{z}=d-ex^2,\quad (abce>0)$	d	$d=1$
F	$\dot{x}=ay+bz,\quad \dot{y}=cx+dy$ $\dot{z}=ex^2-fz$	c,d	$c=-1, d=0.5$
G	$\dot{x}=ax+bz,\quad \dot{y}=cxz+dy$ $\dot{z}=-ex+fy,\quad (be>0)$	a,d	$a=0.4, d=-1$
H	$\dot{x}=ay+bz^2,\quad \dot{y}=cx+dy$ $\dot{z}=ex-fz$	a,d	$a=-1, d=0.5$
K	$\dot{x}=axy-bz,\quad \dot{y}=cx-dy$ $\dot{z}=ex+fz,\quad (be>0)$	d,f	$d=1, f=0.3$
M	$\dot{x}=-az,\quad \dot{y}=-bx^2-cy$ $\dot{z}=d+ex+fy$	d,e	$d=e=1.7$
O	$\dot{x}=ay,\quad \dot{y}=bx-cz$ $\dot{z}=dx+exz+fy$	b,f	$b=1, f=2.7$
P	$\dot{x}=ay+bz,\quad \dot{y}=-cx+dy^2$ $\dot{z}=ex+fy,\quad (be>0)$	a,c	$a=2.7, c=1$
Q	$\dot{x}=-az,\quad \dot{y}=bx-cy$ $\dot{z}=dx+ey^2+fz$	d,f	$d=3.1, f=0.5$
S	$\dot{x}=-ax-by,\quad \dot{y}=cx+dz^2$ $\dot{z}=e+fx$	b,e	$b=4, e=1$
1	$a\dddot{x}+b\ddot{x}-c\dot{x}^2+dx=0$	b	$b=2.017$
2	$a\dddot{x}+b\dot{x}-cx^2+d=0,$ $(ab>0)$	d	$d=0.025$