How to Compute e^{At} ?

(I) Simplest Case — $A^{n \times n}$ is diagonalizable

Suppose A has n linearly independent eigenvectors $v_1, v_2, v_3, \cdots v_n$

corresponding to n eigenvalues, $\lambda_1, \lambda_2, \cdots, \lambda_n$ (which might be repeating)

i.e. $A v_i = \lambda_i v_i, \quad i = 1, 2, \cdots, n$

Then
$$\boxed{e^{At} = P e^{\Lambda t} P^{-1}}$$

(I) Simplest Case — $A^{n\times n}$ is diagonalizable

where $P = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$, $\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

$$\boxed{e^{At} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^{-1}}$$

(II) Jordan Canonical Form

There is a matrix P whose columns v_i are eigenvectors

$$A v_i = \lambda_i v_i \quad \text{i.e.} \quad (A - \lambda_i I) v_i = 0$$

or generalized eigenvectors

$$(A - \lambda_i I)^{k_i} v_i = 0$$

such that

$$\boxed{A = P \Lambda P^{-1}}$$

(II) Jordan Canonical Form

where

$$P = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \boxed{J_1} & & & \\ & \boxed{J_2} & & \\ & & \ddots & \\ & & & \boxed{J_k} \end{bmatrix}$$

$$\begin{bmatrix} J_i \end{bmatrix} = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix} = \begin{bmatrix} \lambda_i & & \\ & \ddots & \\ & & \lambda_i \end{bmatrix} + \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$

Jordan block

$$= \underbrace{\lambda_i I}_{D_i} + N_i$$

(II) Jordan Canonical Form

Note : D_i is diagonal,

N_i is nilpotent, $N_i^{\square} = O$

$D_i N_i = N_i D_i$, i.e. D_i, N_i commute

$$A = P \Lambda P^{-1} \Longrightarrow e^{At} = P e^{\Lambda t} P^{-1}$$

$$e^{\Lambda t} = \begin{bmatrix} \left[e^{J_1 t}\right] & & & \\ & \left[e^{J_2 t}\right] & & \\ & & \ddots & \\ & & & \left[e^{J_k t}\right] \end{bmatrix}$$

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$$e^{J_i t} = I + \frac{(J_i t)}{1!} + \frac{(J_i t)^2}{2!} + \frac{(J_i t)^3}{3!} + \cdots \cdots$$

(II) Jordan Canonical Form

$$e^{J_i t} = I + \frac{(J_i t)}{1!} + \frac{(J_i t)^2}{2!} + \frac{(J_i t)^3}{3!} + \cdots \cdots$$

"Easy" to compute J_i^m

$$J_i^m = (D_i + N_i)^m$$

$$= D_i^m + \binom{m}{1} D_i^{m-1} N_i + \binom{m}{2} D_i^{m-2} N_i^2$$

$$+ \cdots\cdots + \binom{m}{k-1} D_i^{m-k+1} N_i^{k-1}$$

$$\underline{(N_i^k = 0)}$$

(III) Laplace Transform

$$\mathcal{L}(e^{At}) = \int_0^\infty e^{At} e^{-st} \, dt$$

$$= (sI - A)^{-1}$$

$$e^{At} = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)$$

Examples

① $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$,

$$A \overbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}^{v_1} = \overbrace{(-1)}^{\lambda_1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{v_2} = \underbrace{(-3)}_{\lambda_2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & \\ & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

$$e^{At} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & \\ & e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} e^{-t}+e^{-3t} & e^{-t}-e^{-3t} \\ e^{-t}-e^{-3t} & e^{-t}+e^{-3t} \end{bmatrix}$$

② $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$:

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \underset{V_1}{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}} = \underset{\lambda_1}{1} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \underset{V_2}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} = \underset{\lambda_2}{1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} \underset{V_3}{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}} = \underset{\lambda_3}{3} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}^{-1}$$

$$e^{At} = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} e^{t} & & \\ & e^{t} & \\ & & e^{3t} \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -e^{t} & 0 & e^{3t} \\ 2e^{t} & 0 & 0 \\ 0 & e^{t} & 2e^{3t} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -2 & -1 & 1 \\ 1 & \frac{1}{2} & 0 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{3t} & -\frac{1}{2}e^{t}+\frac{1}{2}e^{3t} & 0 \\ 0 & e^{t} & 0 \\ -2e^{t}+2e^{3t} & -e^{t}+e^{3t} & e^{t} \end{bmatrix}$$

③ $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

$$\mathcal{L}(e^{At}) = (sI - A)^{-1} = \begin{pmatrix} s+2 & -1 \\ -1 & s+2 \end{pmatrix}^{-1}$$

$$= \frac{1}{(s+2)^2-1} \begin{bmatrix} s+2 & 1 \\ 1 & s+2 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s+2}{(s+3)(s+1)} & \dfrac{1}{(s+3)(s+1)} \\ \dfrac{1}{(s+3)(s+1)} & \dfrac{s+2}{(s+3)(s+1)} \end{bmatrix}$$

Partial fraction

$$\mathcal{L}(e^{At}) = \begin{bmatrix} \frac{1}{2}\frac{1}{(S+3)} + \frac{1}{2}\frac{1}{(S+1)} & \frac{1}{2}\frac{1}{(S+1)} - \frac{1}{2}\frac{1}{(S+3)} \\ \frac{1}{2}\frac{1}{(S+1)} - \frac{1}{2}\frac{1}{(S+3)} & \frac{1}{2}\frac{1}{(S+3)} + \frac{1}{2}\frac{1}{(S+1)} \end{bmatrix}$$

$\mathcal{L}^{-1}$

$$\boxed{e^{At} = \begin{bmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} \\ \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{bmatrix}}$$