Invariant Subspaces of $A^{n\times n}$

Suppose A is diagonalizable.
Then there are n linearly independent eigenvectors v_i, with eigenvalues λ_i, i.e.

$$\boxed{A v_i = \lambda_i v_i \qquad 1 \le i \le n}$$

&

$$\boxed{\mathbb{R}^n = \operatorname{Span}\{v_1\} \oplus \cdots \oplus \operatorname{Span}\{v_i\}}$$

Invariant Subspaces of $A^{n\times n}$

Suppose A is diagonalizable.

Solution of

$$\boxed{\frac{dX}{dt} = AX,} \qquad X(0) = X_0$$

is given by

$$\boxed{X(t) = c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_n t} v_n}$$

where

$$X_0 = c_1 v_1 + \cdots + c_n v_n$$

initial condition

Invariant Subspaces of $A^{n \times n}$

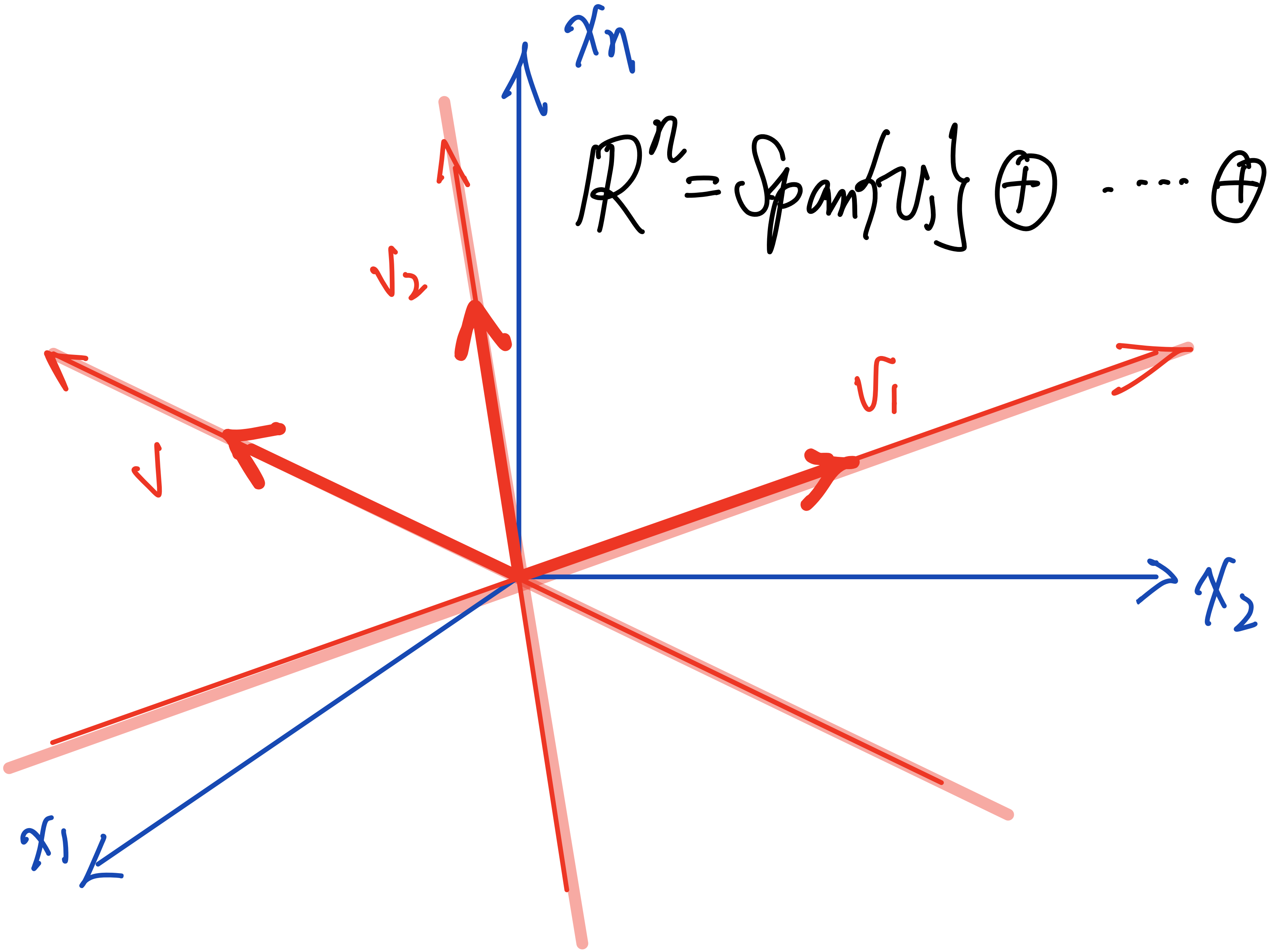
Suppose A is diagonalizable.

Note: if $x_0 = c_i v_i \in \mathrm{Span}\{v_i\}$

then $x(t) = c_i e^{\lambda_i t} v_i \in \mathrm{Span}\{v_i\}$

i.e. $\boxed{\text{Span}\{v_i\} \text{ is invariant under the dynamics}}$

Invariant Subspaces of $A^{n\times n}$



$$\mathbb{R}^n = \mathrm{Span}\{v_1\} \oplus \cdots\cdots \oplus \mathrm{Span}\{v_i\}$$

Invariant Subspaces of $A^{n\times n}$

More generally, [M, Sec. 2.6]

- $p(\lambda) = \det(A-\lambda I)$ characteristic poly.

$$= c(\lambda-\lambda_1)^{m_1}(\lambda-\lambda_2)^{m_2}\cdots(\lambda-\lambda_k)^{m_k}$$

- $E_i = \mathrm{Null}\{(A-\lambda_i I)^{m_i}\}$ $\dim E_i = m_i$

$$= \{u : (A-\lambda_i I)^{m_i}u = 0\}$$

$=$ Generalized eigenspace

Invariant Subspaces of $A^{n\times n}$

More generally,

- $\boxed{\mathbb{R}^n = E_1 \oplus E_2 \oplus \cdots \oplus E_k}$

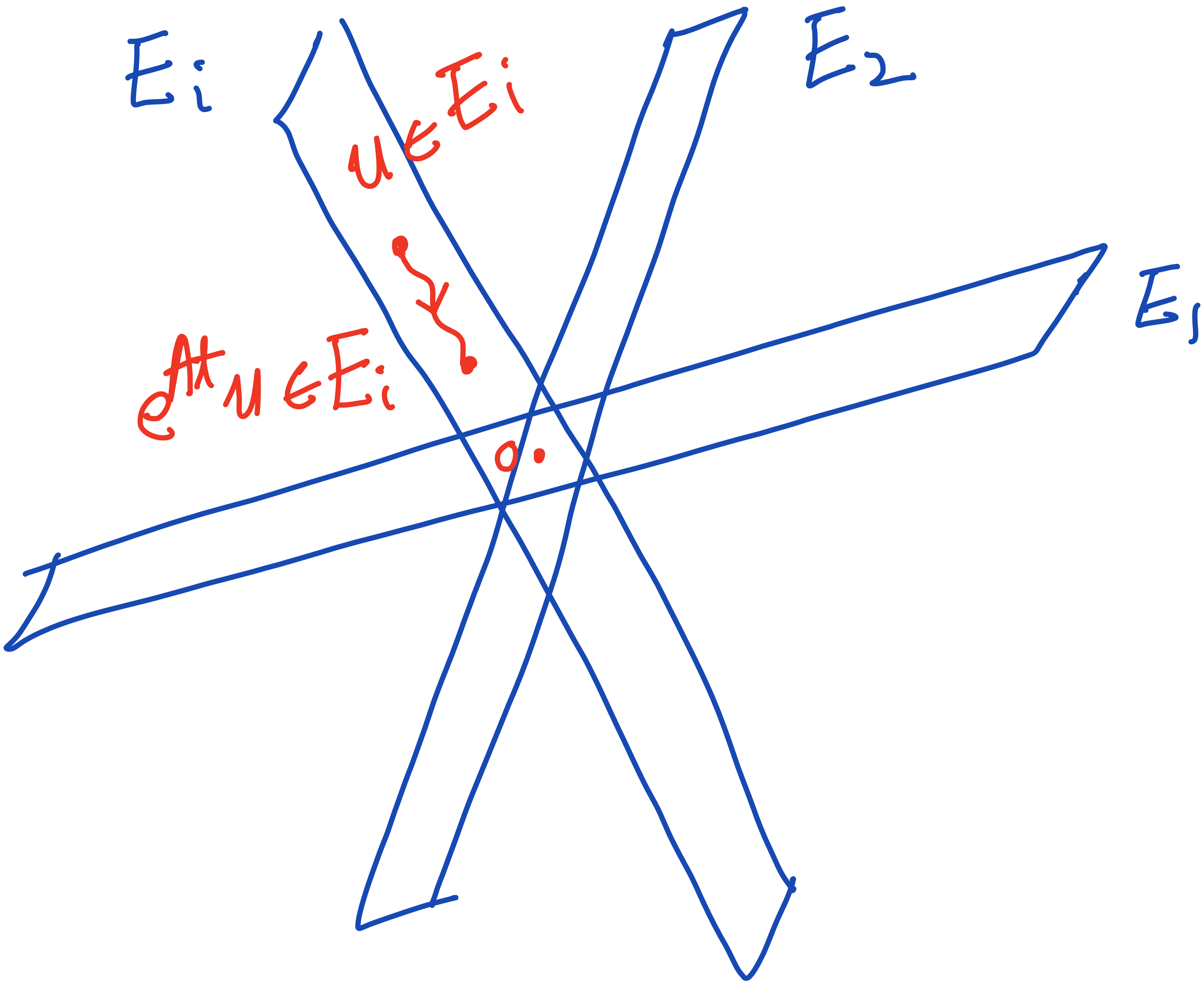
- E_i is invariant under A,

  i.e. $u \in E_i$ then $Au \in E_i$

- E_i is invariant by e^{At},

  i.e. $u \in E_i$, then $e^{At}u \in E_i$

Invariant Subspaces of $A^{n\times n}$



Invariant Subspaces of $A^{n \times n}$

[M, Sec 2.7]

$$\mathbb{R}^n = E_1 \oplus E_2 \oplus \cdots\cdots \oplus E_k$$

$$= E_s \oplus E_c \oplus E_u$$

where $E_s = \bigoplus_{Re(\lambda_i)<0} E_{\lambda_i}$ = stable subspace

$E_u = \bigoplus_{Re(\lambda_i)>0} E_{\lambda_i}$ = unstable subspace

$E_c = \bigoplus_{Re(\lambda_i)=0} E_i$ = center subspace

Linear Stability

[M, Sec 2.7]
[Bellman, p.25]

- $X_0 \in E_s \iff$ there is $K>0$ s.t

$$\|e^{At}X_0\| \leq C e^{-Kt}\|X_0\| \text{ for all } t>0$$

i.e. $e^{At}X_0 \xrightarrow{t\to+\infty} 0$ exponentially fast in t

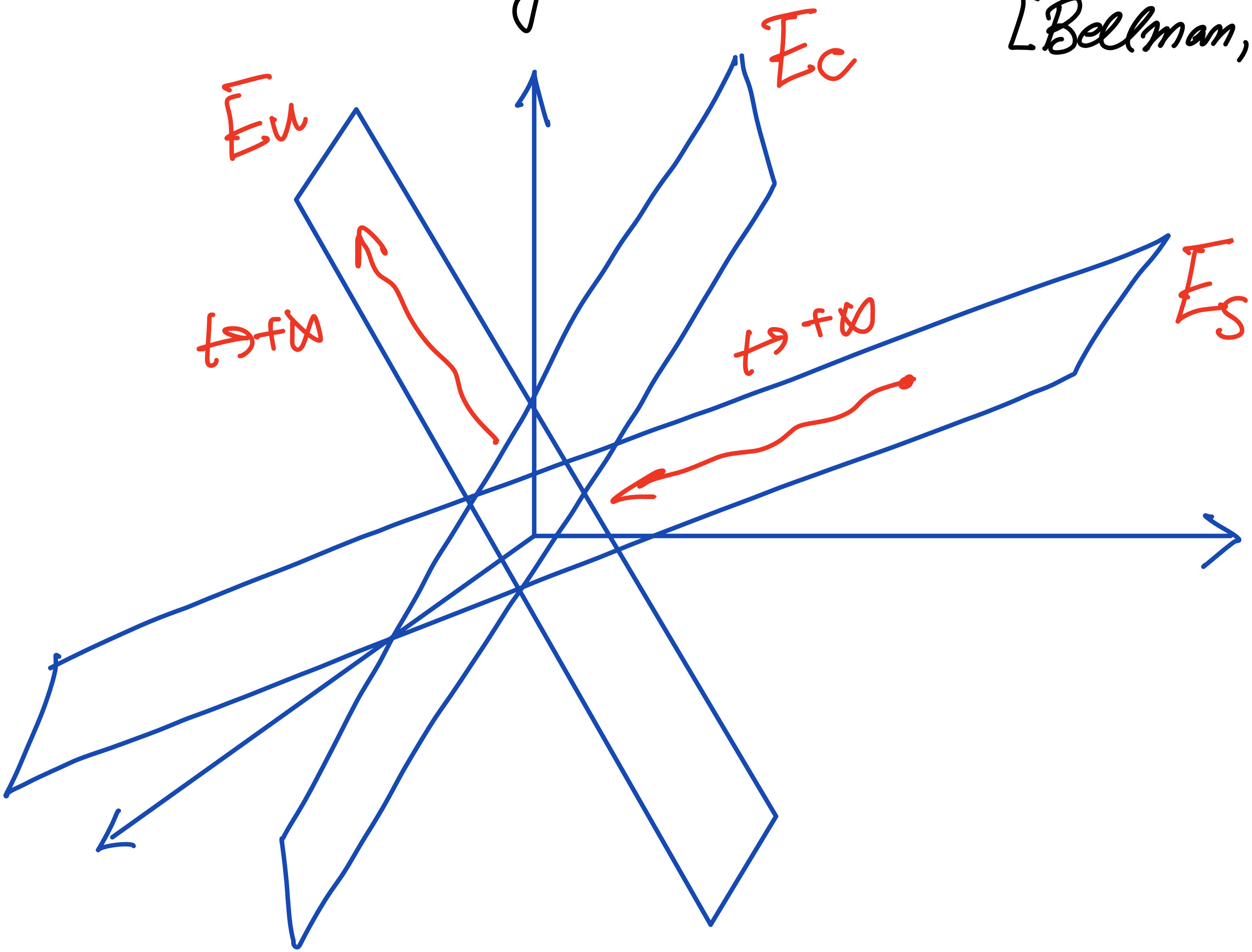
- $X_0 \in E_u \iff$ there is $K>0$ s.t

$$\|e^{At}X_0\| \leq C e^{Kt}\|X_0\| \text{ for all } t<0$$

i.e. $e^{At}X_0 \xrightarrow{t\to-\infty} 0$ exponentially fast in t

Linear Stability

[M, Sec 2.7]
[Bellman, p.25]



Linear Stability [M, Sec 2.7]

- But it is not true that if $X_0 \in E_c$, then $\|e^{At}X_0\|$ remain bounded.

e.g. $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0 + y_0 t \\ y_0 \end{pmatrix} \longrightarrow \infty \quad \text{if } y_0 \neq 0$$

Gronwall's Inequality [M, p.95, p.103 Ex.9]

If $$g(t) \leq C + \int_0^t k(s)g(s)\,ds$$

with $k(t) \geq 0$,

then $$g(t) \leq C e^{\int_0^t k(s)\,ds} \quad (t \geq 0)$$

Gronwall's Inequality [M, p.95, p.103 Ex.9]

If $$g(t) \leq C + \int_0^t k(s) g(s)\,ds$$

with $k(t) \geq 0$,

then $$g(t) \leq C\, e^{\int_0^t k(s)\,ds} \quad (t \geq 0)$$

(If $C = 0$ $\Longrightarrow$ $g(t) \leq 0$

If $C = 0$, $g(t) \geq 0$ $\Longrightarrow$ $g(t) \equiv 0$.)

Linear Stability (Perturbative Statements)

If all the solution of $\dfrac{dX}{dt} = AX$ goes to zero as $t \to +\infty$,

then so is the solution of

$$\frac{dX}{dt} = AX + h(t), \quad \text{provided}$$

$$\|h(t)\| \leq C e^{-\delta t}$$

Linear Stability (Perturbative Statements)

If all the solution of $\dfrac{dX}{dt} = AX$ goes to zero as $t \to +\infty$,

then so is the solution of

$$\frac{dX}{dt} = AX + h(t), \quad \text{provided}$$

or $\displaystyle\int_0^\infty \|h(t)\| dt < \infty$, or $\displaystyle\lim_{t\to+\infty} \|h(t)\| = 0$

Linear Stability (Perturbative Statements)

Thm 1 (Bellman, p.34)

If all the solutions of $\frac{dX}{dt} = AX$ are bounded for all $t > 0$,

then the same is true for the solution of $\frac{d}{dt}X = (A + B(t))X$, provided

$$\int_0^\infty \|B(s)\|\, ds < \infty$$

Linear Stability (Perturbative Statements)

Thm 2 (Bellman, p.36)

If all the solutions of $\frac{dX}{dt} = AX$ go to zero as $t \longrightarrow +\infty$,

then the same is true for the solution of $\frac{d}{dt}X = (A+B(t))X$, provided

$$\int_0^\infty \|B(s)\|\,ds < \infty$$

Linear Stability (Perturbative Statements)

Thm 2' (Bellman, p.36)

If all the solutions of $\frac{dX}{dt} = AX$ go to zero as $t \longrightarrow +\infty$,

then the same is true for the solution of $\frac{d}{dt}X = (A + B(t))X$, alternatively if

i.e. $\|B(t)\| < \varepsilon(t) \ll 1$ for $t > t_0$

Linear Stability (Perturbative Statements)

Thm 2' (Bellman, p.36)

If all the solutions of $\frac{dX}{dt} = AX$
go to zero as $t \longrightarrow +\infty$,
then the same is true for the solution
of $\frac{d}{dt}X = (A + B(t))X$, alternatively if

$$\lim_{t\to\infty} \|B(t)\| = 0$$

Linear Stability (Perturbative Statements)

Thm 5 (Bellman, p.42)

There is an $A(t)$ s.t. all the solutions of $\frac{dX}{dt} = A(t)X$ go to zero as $t \to +\infty$

and $B(t)$ satisfying $\int_0^\infty \|B(t)\| \, dt < \infty$

and yet all (non-trivial) solutions of

$\frac{dX}{dt} = \big(A(t) + B(t)\big)X$ go to infinity as $t \to +\infty$