Length (Norm) of Vectors and Matrices

$X \in \mathbb{R}^n, \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$\|\cdot\| : \ X \in \mathbb{R}^n \longrightarrow \|X\|$

① $\|X\| \geq 0, \quad = 0 \iff X = \vec{0}$

② $\|cX\| = |c| \, \|X\|$

③ $\|X+Y\| \leq \|X\| + \|Y\|$

Triangular Inequality

Length (Norm) of Vectors and Matrices

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Examples of $\|\cdot\|$

"length"

① $\|X\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

② $\|X\|_1 = |x_1| + |x_2| + \cdots + |x_n|$

③ $\|X\|_p = \left(|x_1|^p + |x_1|^p + \cdots + |x_n|^p\right)^{1/p}$

④ $\|X\|_\infty = \max_i |x_i|$

Length (Norm) of Vectors and Matrices

$$X \in \mathbb{R}^n, \qquad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

$(p=2)$ **Inner product:**

$$\langle X, Y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

① $\langle X, Y \rangle = \langle Y, X \rangle$

② $\langle \alpha X + \beta Y, Z \rangle = \alpha \langle X, Z \rangle + \beta \langle Y, Z \rangle$

③ $\langle X, X \rangle = \|X\|_2^2 \geq 0, \quad = 0 \iff X = 0$

Length (Norm) of Vectors and Matrices

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$(p=2)$ Inner product:

$$\langle X, Y \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Cauchy-Schwarz Ineq:

$$|\langle X, Y \rangle| \leq \|X\| \|Y\|$$

$=$ holds if and only if $X \parallel Y$

$X = \alpha Y$ or $Y = \alpha X$

Length (Norm) of Vectors and Matrices

$$A = (a_{ij})$$

$$\|A\|_p = \begin{cases} \left(\sum_{i,j} |a_{ij}|^p \right)^{\frac{1}{p}}, & p = 1, 2, \cdots \\ \max_{i,j} |a_{ij}| & p = \infty \end{cases}$$

① $\|A\| \geq 0, \quad = 0 \iff A = 0$

② $\|cA\| = |c| \, \|A\|$

③ $\|A+B\| \leq \|A\| + \|B\|$

Length (Norm) of Vectors and Matrices

$$\|AX\|_{p_1} \leq \underline{C(n, p_1, p_2, p_3)} \|A\|_{p_2} \|X\|_{p_3}$$

$$\|AB\|_{p_1} \leq \underline{C(n, p_1, p_2, p_3)} \|A\|_{p_3} \|B\|_{p_2}$$

Examples:

$$\|AX\|_1 \leq \begin{cases} \|A\|_1 \|X\|_\infty \leq \|A\|_1 \|X\|_1 \\ n\|A\|_\infty \|X\|_1 \end{cases}$$

$$\|AB\|_1 \leq n \|A\|_\infty \|B\|_1$$

Length (Norm) of Vectors and Matrices

Operator norm of a matrix:

$$\|A\|_{op,p} = \max_{X \neq 0} \frac{\|AX\|_p}{\|X\|_p}$$

$$= \max_{\|X\|=1} \|AX\|_p$$

$$\|AX\|_p \leq \|A\|_{op,p} \|X\|_p$$

Length (Norm) of Vectors and Matrices

In fact, in finite dimensions,
all norms are equivalent:

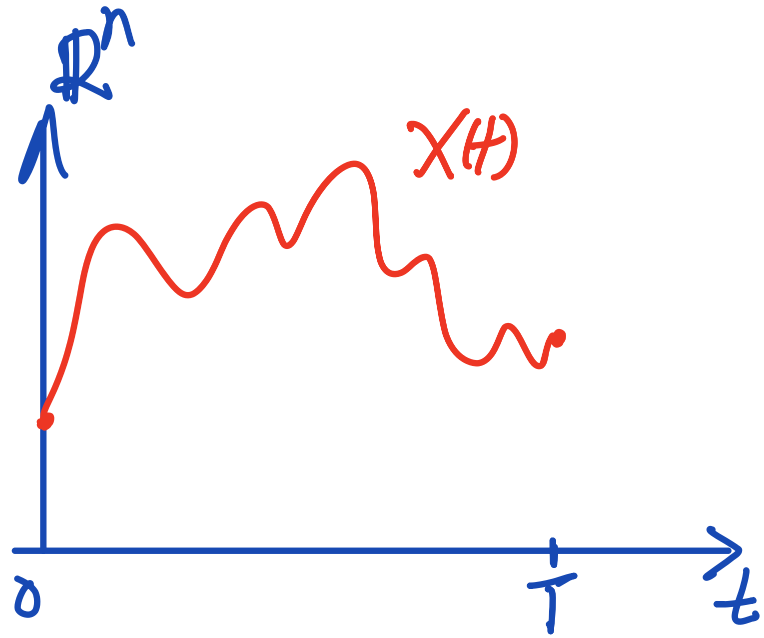
$$\boxed{C_1(n,p,q)\,\|\cdot\|_q \le \|\cdot\|_p \le C_2(n,p,q)\,\|\cdot\|_q}$$

Note: The constants depend on dimensions.

Function Spaces

$$X : [0,T] \longrightarrow \mathbb{R}^n$$

$$(t \in [0,T] \longrightarrow X(t) \in \mathbb{R}^n)$$



$\boxed{C^0([0,T];\mathbb{R}^n) : \text{space of continuous functions}}$

$$X \in C^0([0,T];\mathbb{R}^n) : \quad \|X\|_{C^0} = \sup_{t\in[0,T]} \|X(t)\|$$

$$X, Y \in C^0([0,T];\mathbb{R}^n) : \quad \|X - Y\|_{C^0} = \sup_{t\in[0,T]} \|X(t) - Y(t)\|$$

Function Spaces

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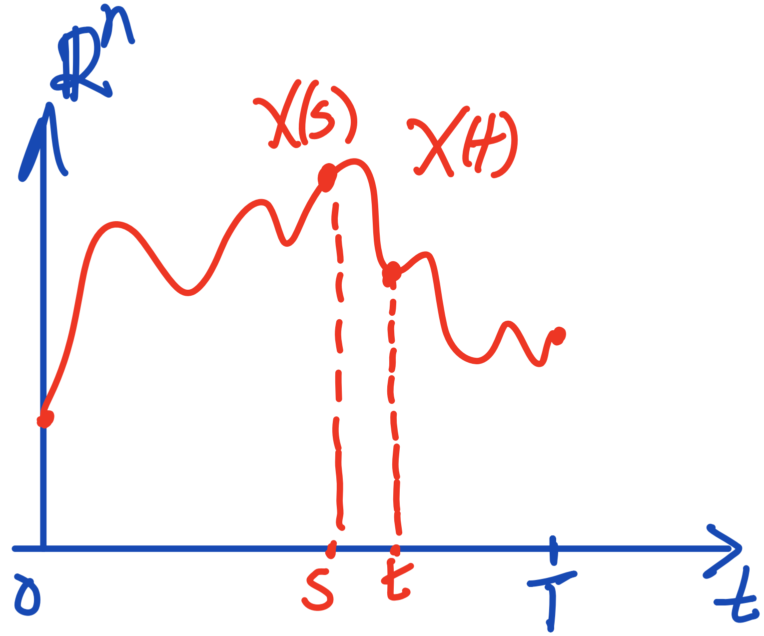

$\boxed{C^0([0,T]; \mathbb{R}^n) : \text{space of continuous functions}}$

X is continuous at t if $\forall \varepsilon > 0$, $\exists\, \delta(t) > 0$

such that $\forall s$ satisfying $|t-s| \leq \delta(t)$,

then $\|X(t) - X(s)\| \leq \varepsilon$

Function Spaces

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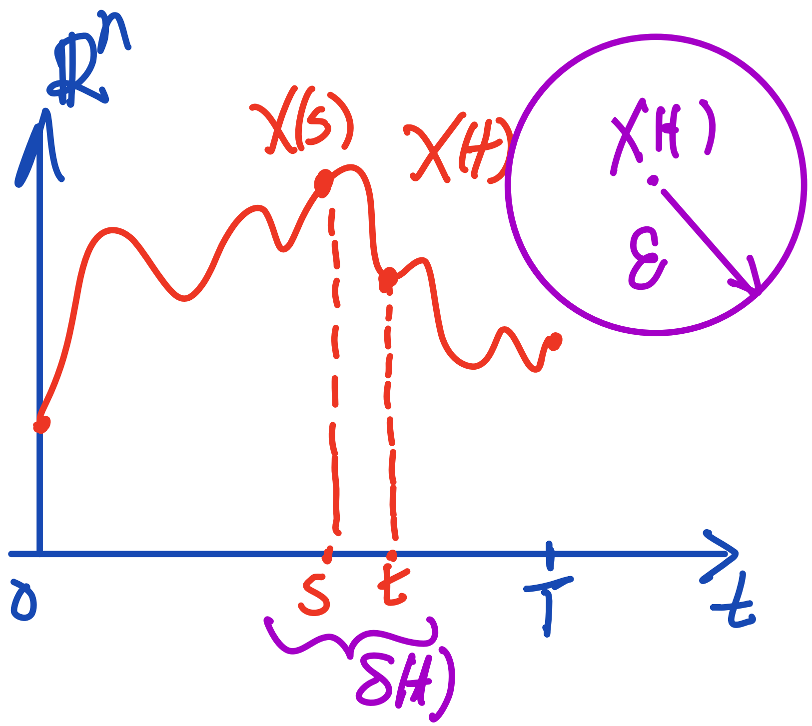
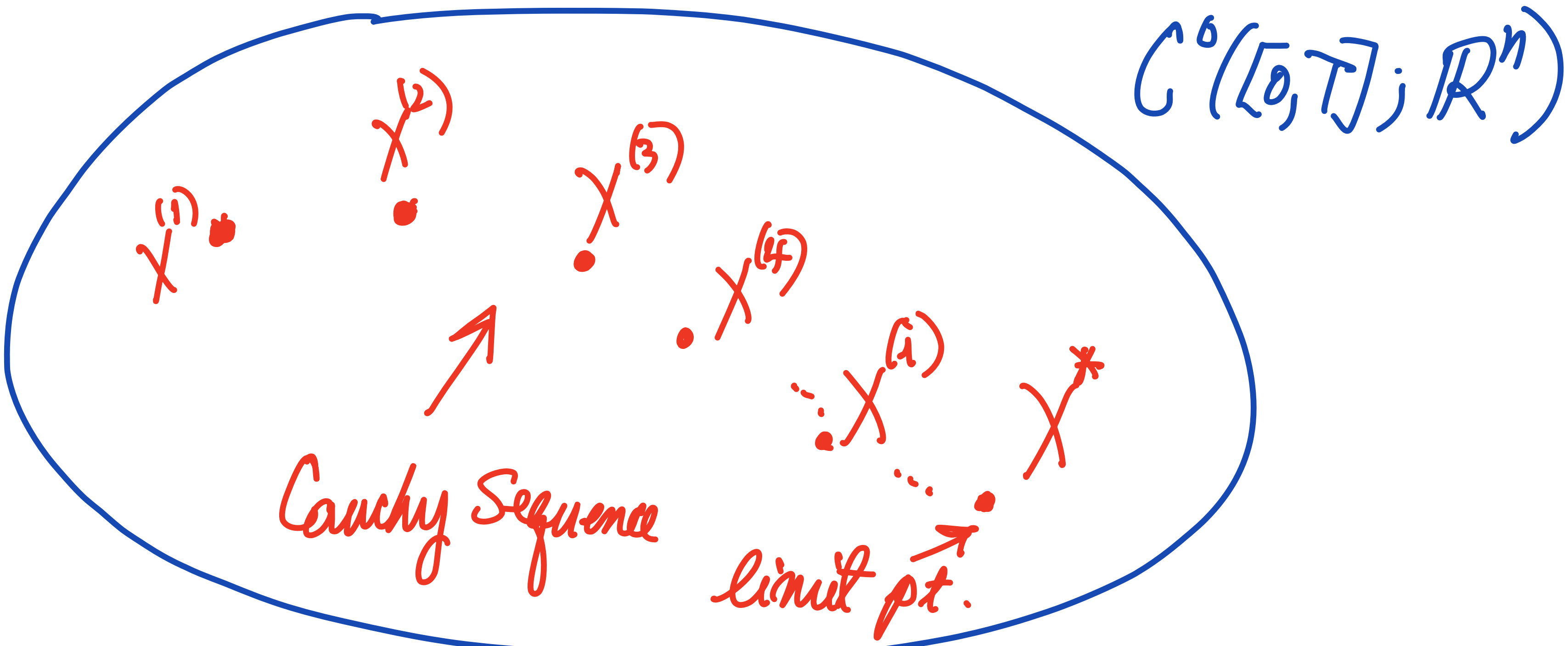


$$\|X(s) - X(t)\| \le \varepsilon$$
$$\left(X(s) \in B_{X(t)}(\varepsilon) \right)$$

Function Spaces — Completeness

$C^0([0,T];\mathbb{R}^n)$ is complete in the following sense: any Cauchy Sequence converges



Function Spaces — Completeness

$C^0([0,T];\mathbb{R}^n)$ is complete in the following sense:

any Cauchy Sequence converges

i.e. Given $\{X^{(n)}\}_{n=1,2,\cdots}$

if $\forall \varepsilon$, $\exists N>0$ s.t. $\forall n,m \geq N$, it holds that

$$\|X^{(n)}-X^{(m)}\|_{C^0} \leq \varepsilon$$

then there exists an $X^* \in C^0([0,T];\mathbb{R}^n)$ s.t.

$X^{(n)} \longrightarrow X^*$ in C^0, i.e. $\lim_{n\to\infty} \|X^{(n)}-X^*\|_{C^0}=0$

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Function Spaces — Completeness

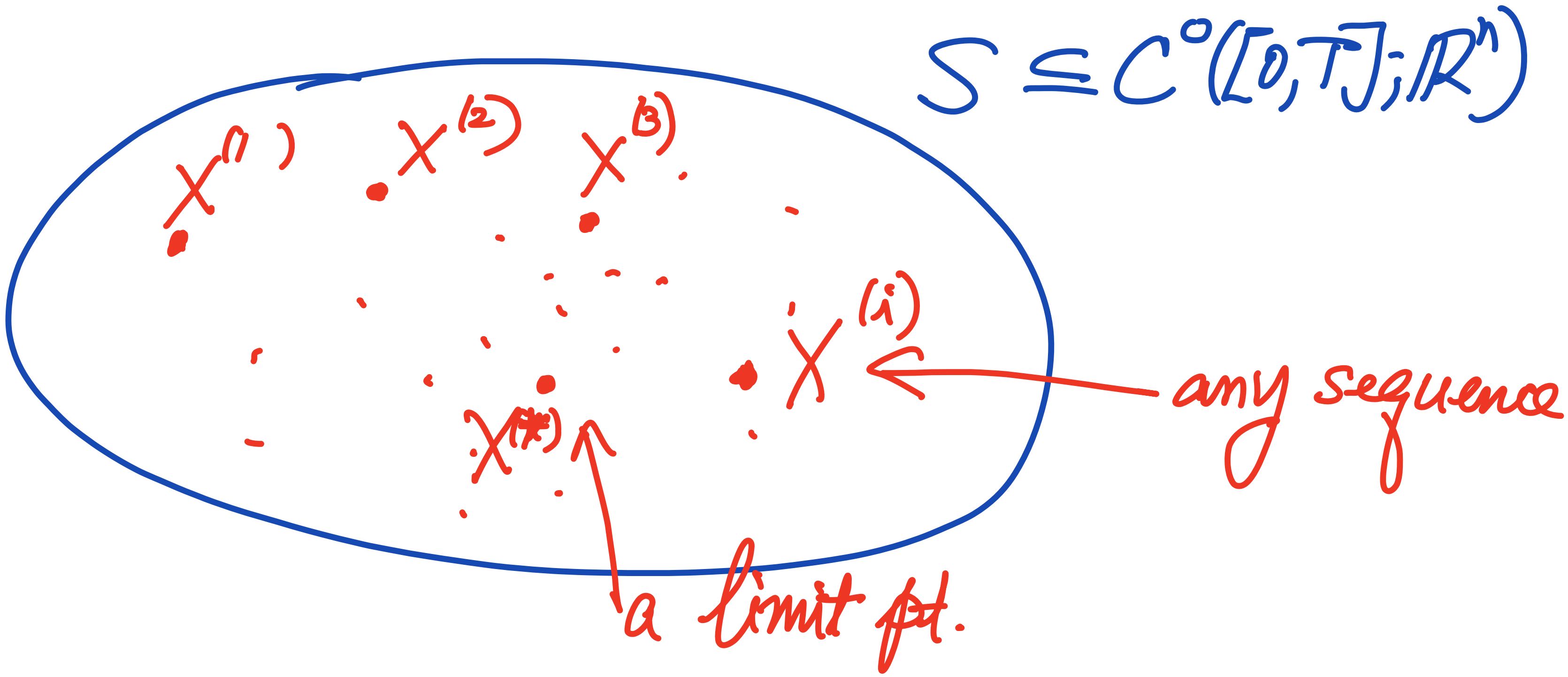
A sufficient condition for $\{X^{(i)}\}_{i=1}^{\infty}$ to has a limit pt:

$$\underbrace{\sum_{i=1}^{\infty} \|X^{(i+1)} - X^{(i)}\|}_{\text{Sum of all the increments } < \infty} < \infty$$

($\Longrightarrow$ then $\{X^{(i)}\}$ is a Cauchy seq.)

Function Spaces — Compactness

A subset $S \subseteq C^0([0,T];\mathbb{R}^n)$ is compact if any sequence in S has a convergent subsequence with limit pt. inside S.



Function Spaces — Compactness

A subset $S \subseteq C^0([0,T];\mathbb{R}^n)$ is compact

if $\forall \{X^{(n)}\}_{n=1,2,\cdots}$, $\exists \{X^{(n_k)}\}_{k=1,2,\cdots}$ s.t.

(sequence) (subsequence)

$X^{(n_k)}$ converges in S as $k \to +\infty$

i.e. $\exists X^* \in C^0([0,T];\mathbb{R}^n)$ s.t.

$$X^{n_k} \xrightarrow{k\to\infty} X^* \quad \left(\lim_{k\to\infty} \|X^{(n_k)} - X^*\|_{C^0} = 0\right)$$

Function Spaces — Compactness

Criterion for Compactness (Arzela-Ascoli Thm)

$$\boxed{S \subseteq C^0([0,T];\mathbb{R}^n) \text{ is compact}}$$

$$\Updownarrow$$

S is bounded and equi-continuous:

(1) $\exists M$ s.t. $\|X\|_{C^0} \leq M \quad \forall X \in S$

(2) $\forall \varepsilon, \exists \delta > 0$ s.t. $\forall X \in S, \ \forall t, s, \ |t-s| \leq \delta$
it holds that: $\|X(t) - X(s)\| \leq \varepsilon$

Function Spaces — Compactness

Criterion for Compactness (Arzela-Ascoli Thm)

$$\boxed{S \subseteq C^0([0,T];\mathbb{R}^n) \text{ is compact}}$$

$\Updownarrow$ choice of δ does not depend on t

S is bounded and equi-continuous:

(1) $\exists M$ s.t. $\|X\|_{C^0} \le M \quad \forall X \in S$

(2) $\forall \varepsilon, \exists \delta > 0$ s.t. $\forall X \in S$, $\forall t, s$, $|t-s| \le \delta$ it holds that: $\|X(t) - X(s)\| \le \varepsilon$

Function Spaces — Compactness

Criterion for Equi-continuity

S is "uniformly" Lipschitz:

$$\boxed{\begin{array}{l}\exists K>0 \text{ s.t. } \forall X\in S,\ X \text{ is } K\text{-Lipschitz} \\ \text{i.e. } \quad \|X(t)-X(s)\| \leq K|t-s|\end{array}}$$

($\forall \varepsilon$, set $\delta = \frac{\varepsilon}{K}$. Then if $|t-s|\leq\delta$, then $\|X(t)-X(s)\|\leq\varepsilon$)