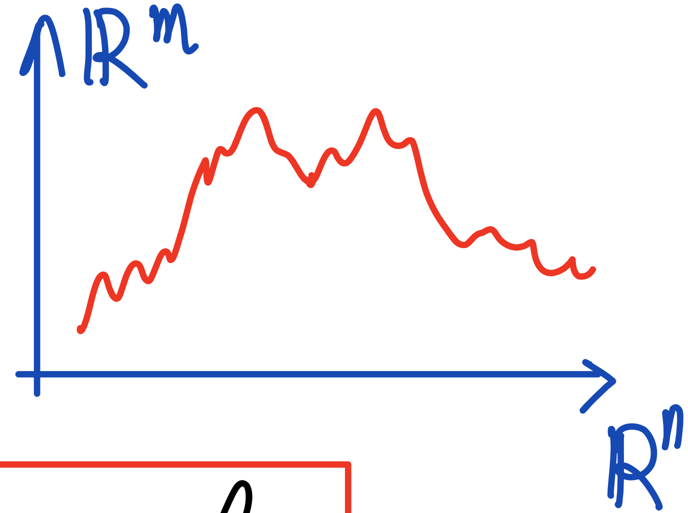
Regularity (Smoothness) Property of Functions

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$



f is continuous at x,

if $y \longrightarrow x$, then $f(y) \longrightarrow f(x)$

Mathematically,

$\forall \varepsilon > 0, \exists\, \delta(x) < 0$, such that

if $|y - x| < \delta(x)$, then $|f(y) - f(x)| \leq \varepsilon$

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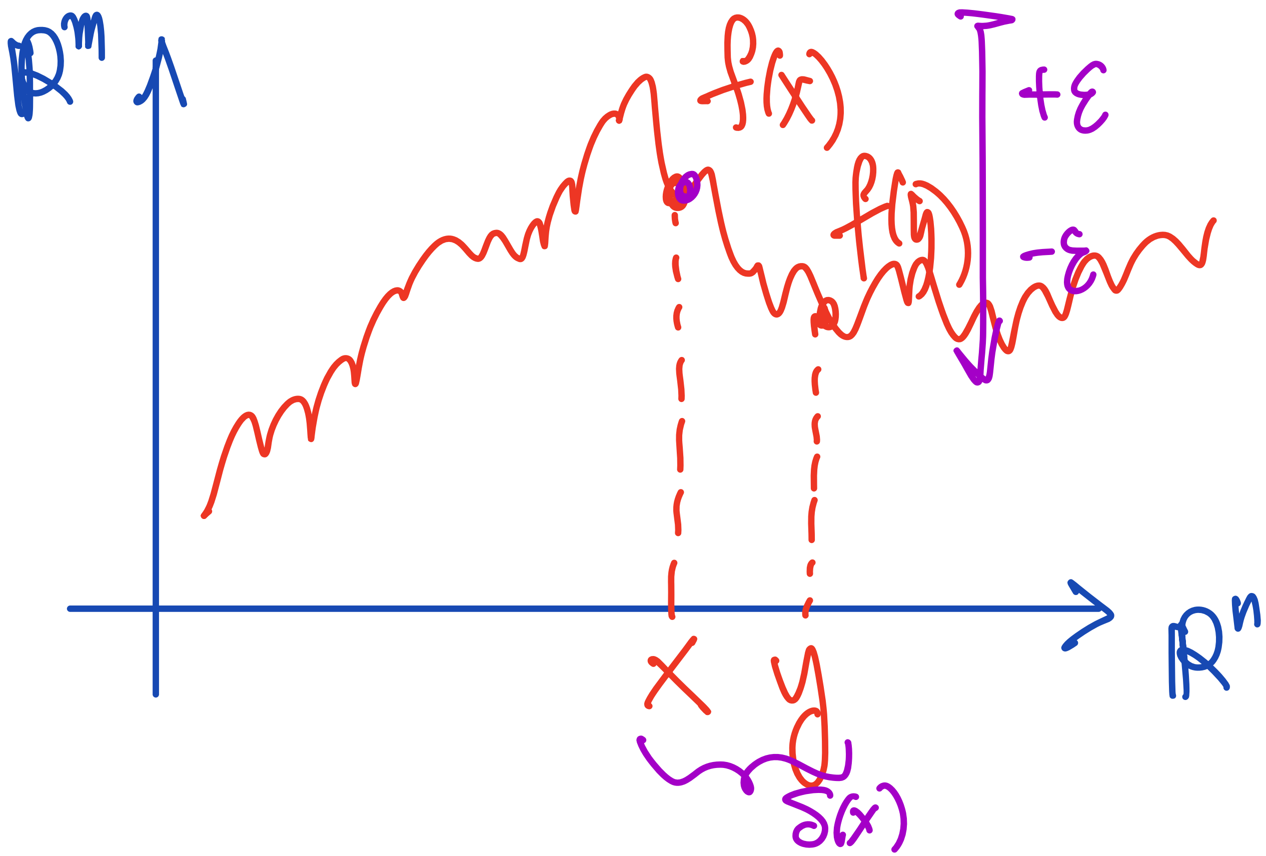

Regularity (Smoothness) Property of Functions

$$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

f is uniformly continuous

if the rate of convergence of $f(y) \to f(x)$ does not depend on x.

Mathematically,

$\forall \varepsilon > 0$, $\exists \delta < 0$, such that

if $|y - x| < \delta$, then $|f(y) - f(x)| \leq \varepsilon$

Regularity (Smoothness) Property of Functions

$$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

f is locally Lipschitz at x,
if there is a K_x and δ_x s.t.

$$\boxed{\text{if } |y-x| \le \delta_x, \text{ then } |f(y)-f(x)| \le K_x |x-y|}$$

f is globally Lipschitz, if there is $K > 0$
s.t.

$$\boxed{|f(y)-f(x)| \le K |y-x|}$$

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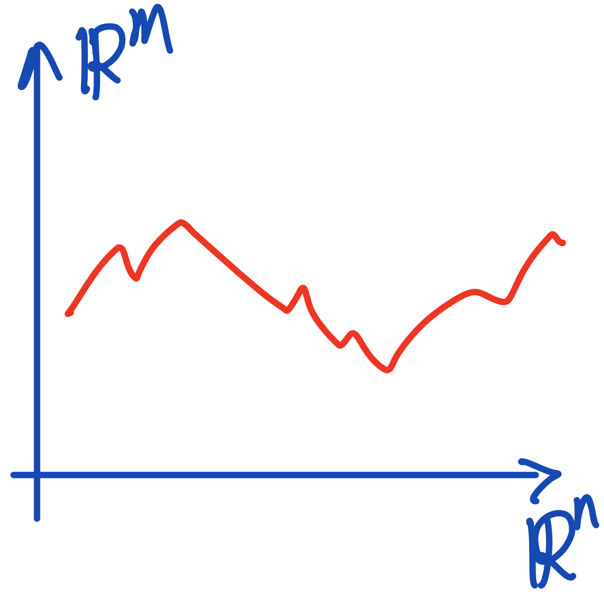
$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

Lipschitz $\Longleftrightarrow$ Slope or rate of change of f is bounded (by a constant)

$$\boxed{\frac{\|f(x)-f(y)\|}{\|x-y\|} \leq K}$$



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Note: $\boxed{\text{Lipschitz with constant } K \Longrightarrow \text{Uniform continuity}}$

$\forall \varepsilon > 0$, let $\delta = \frac{\varepsilon}{K}$. Then for all $\|y-x\| \le \frac{\delta}{K}$,

$$\boxed{\|f(y)-f(x)\| \le K\|y-x\| \le K\frac{\varepsilon}{K} = \varepsilon}$$

Regularity (Smoothness) Property of Functions

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

f is differentiable at x, if there is a matrix $A^{n\times m}$ such that

$$\boxed{\lim_{y \to x} \frac{\| f(y) - f(x) - A(y-x) \|}{\| y - x \|} = 0}$$

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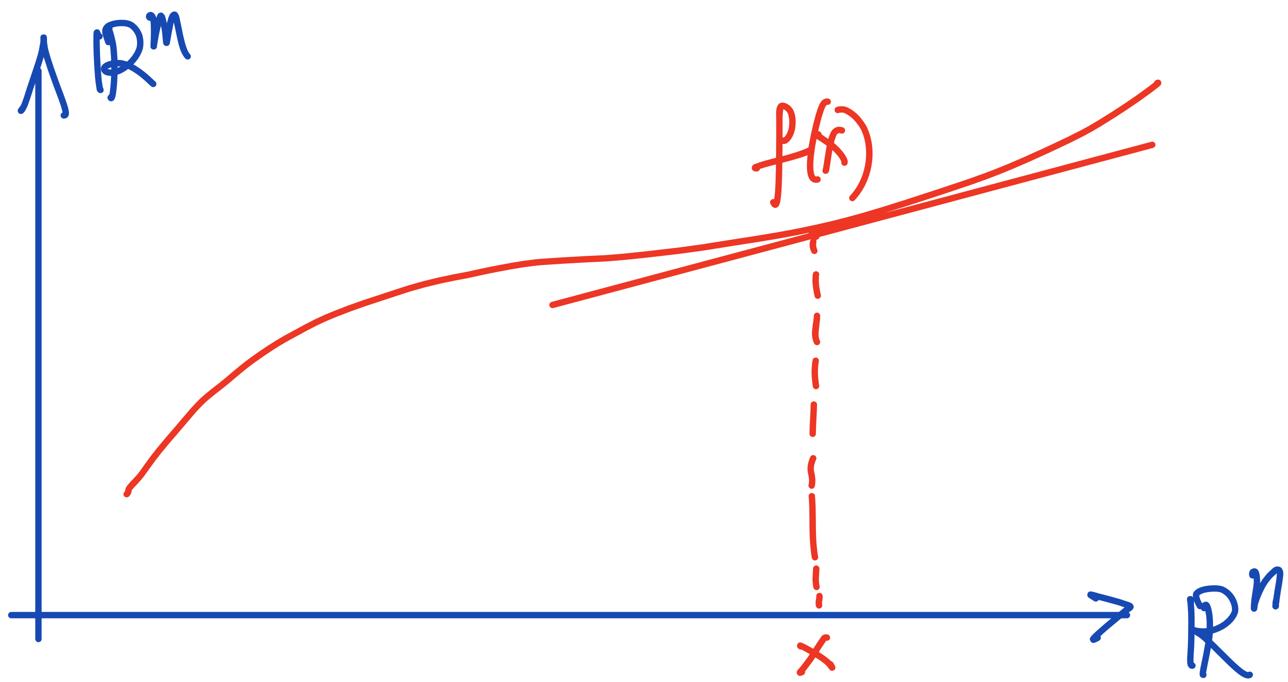

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f is differentiable at x, if there is a matrix $A^{n\times m}$ such that

$$\boxed{\begin{array}{l} \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. if } \|y - x\| \leq \delta, \text{ then} \\ \quad \|f(y) - f(x) - A(y-x)\| \leq \varepsilon \|y - x\| \end{array}}$$

Regularity (Smoothness) Property of Functions

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

f is differentiable at x, if there is a matrix $A^{n\times m}$ such that

$$\boxed{\lim_{h \to 0} \frac{\| f(x+h) - f(x) - Ah \|}{\| h \|} = 0}$$

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Regularity (Smoothness) Property of Functions

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

f is differentiable at x, if there is a matrix $A^{n \times m}$ such that

$$\boxed{f(y) = f(x) + A(y-x) + o(\|y-x\|)}$$

$$\lim_{a \to 0} \frac{o(a)}{a} = 0 \quad \text{or} \quad o(a) \ll a$$

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Regularity (Smoothness) Property of Functions

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$\boxed{A = [Df](x)}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \longrightarrow f(x) = \begin{pmatrix} f_1(x_1, \cdots x_n) \\ f_2(x_2, \cdots x_n) \\ \vdots \\ f_m(x_1, \cdots x_n) \end{pmatrix}$$

Regularity (Smoothness) Property of Functions

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$\boxed{A = [Df](x)}$$

$$[Df](x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Regularity (Smoothness) Property of Functions

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$\boxed{A = [Df](x)}$$

$$[Df](x) = \begin{array}{cc} & m \times n \\ \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \\ \dfrac{\partial f_m}{\partial x_1} & \dfrac{\partial f_m}{\partial x_2} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix} & \begin{array}{l} \leftarrow \nabla f_1 \\ \\ \leftarrow \nabla f_2 \\ \vdots \\ \leftarrow \nabla f_m \end{array} \end{array}$$