C^1-Dependence of Solution in Initial Condition & Parameters

Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be C^1-function.

i.e. $D_x F(x)$ exists and is continuous

Consider

$$\boxed{\begin{array}{l} \dfrac{dX}{dt} = F(X) \,; \quad X(0) = y \\[2em] X = X(t;y), \quad X(0;y) = y \end{array}}$$

C^1- Dependence of Solution in Initial Condition & Parameters

Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be C^1- function.

i.e. $D_x F(x)$ exists and is continuous

Consider $\frac{d}{dt} X(t;y) = F(X(t;y))$

D_y

① $\frac{d}{dt}\left[D_y X(t;y)\right] = \left[D_x F(X(t;y))\right]\left[D_y X(t;y)\right]$

② $X(0;y) = y \Rightarrow \left[D_y X(0;y)\right] = I$

C^1- Dependence of Solution in Initial Condition + Parameters

Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be C^1- function.

i.e. $D_x F(x)$ exists and is continuous

ie. $[D_y X(t;y)]$ is a solution to:

$$\boxed{\frac{d}{dt}\Phi(t) = [D_x F(X(t;y))]\,\Phi(t), \quad \Phi(0) = I} \qquad (*)$$

ie. $[D_y X(t;y)]$ is a fundamental solution to $(*)$

C^1- Dependence of Solution in Initial Condition & Parameters

Let $F: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be C^1- function.

i.e. $D_x F(x)$ exists and is continuous

$[D_x F(X(t;y))] = A(t)$ "is known"

i.e. $[D_y X(t;y)]$ is a solution to:

$$\boxed{\frac{d}{dt}\Phi(t) = [D_x F(X(t;y))]\,\Phi(t), \quad \Phi(0) = I} \qquad (*)$$

i.e. $[D_y X(t;y)]$ is a fundamental solution to $(*)$

C^1– Dependence of Solution in Initial Condition & Parameters

Let $F: \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ be C^1– function.

X α, parameter

$$F: (X,\alpha) \longrightarrow F(X,\alpha)$$

Let $X = X(t;y,\alpha)$ solve

$$\begin{cases} \frac{d}{dt} X(t;y,\alpha) = F(X(t;y,\alpha),\alpha) \\ X(0;y,\alpha) = y \end{cases}$$

C^1- Dependence of Solution in Initial Condition & Parameters

Let $F: \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ be C^1- function.

X α, parameter

$$F: (X, \alpha) \longrightarrow F(X, \alpha)$$

Let $X = X(t; y, \alpha)$ solve

$$D_\alpha \quad \begin{cases} \frac{d}{dt} X(t; y, \alpha) = F(X(t; y, \alpha), \alpha) \\ X(0; y, \alpha) = y \end{cases}$$

C^1- Dependence of Solution in Initial Condition & Parameters

Let $F: \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ be C^1- function.

$$F: (X, \alpha) \longrightarrow F(X, \alpha)$$

$$\frac{d}{dt} X(t; y, \alpha) = F(X(t; y, \alpha), \alpha)$$

D_α

① $$\frac{d}{dt}[D_\alpha X] = [D_X F(X, \alpha)][D_\alpha X] + [D_\alpha F(X, \alpha)]$$

② $$X(0, y, \alpha) = y \Longrightarrow D_\alpha X(0, y, \alpha) = [0]$$

C^1– Dependence of Solution in Initial Condition + Parameters

Let $F : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ be C^1– function.

$$F : (X, \alpha) \longrightarrow F(X, \alpha)$$

ie. $[D_\alpha X(t, y, \alpha)]$ is a solution to:

$$\boxed{\begin{aligned} \frac{d}{dt}[\Psi(t)] &= [D_X F(X)][\Psi(t)] + [D_\alpha F(X, \alpha)] \\ [\Psi(0)] &= [0] \end{aligned}}$$

(homogeneous initial data) (inhomogeneous term)

C^1- Dependence of Solution in Initial Condition + Parameters

Let $F: \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ be C^1- function.

$$F: (X, \alpha) \longrightarrow F(X, \alpha)$$

ie. $[D_\alpha X(t, y, \alpha)]$ is a solution to:

$$\boxed{[\Psi(t)] = \int_0^t \Phi(t)\,\Phi(s)^{-1} D_\alpha F(X(s, \alpha), \alpha)\, ds}$$

C^1– Dependence of Solution in Initial Condition & Parameters

Let $F: \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ be C^1– function.

$$F: (X, \alpha) \longrightarrow F(X, \alpha)$$

Now suppose the initial condition also depends on α:

$$\boxed{\begin{gathered} \frac{dX}{dt} = F(X, \alpha); \quad X(0) = y(\alpha) \\ X = X(t; y(\alpha), \alpha) \end{gathered}}$$

C^1-Dependence of Solution in Initial Condition & Parameters

Let $F: \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ be C^1-function.

$$F: (X, \alpha) \longrightarrow F(X, \alpha)$$

Now suppose the initial condition also depends on α:

$$\boxed{d_\alpha \left[X(t, y(\alpha), \alpha) \right] = ?}$$

C^1- Dependence of Solution in Initial Condition & Parameters

Let $F : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ be C^1- function.

$$F : (X, \alpha) \longrightarrow F(X, \alpha)$$

(M1)

$$\boxed{d_\alpha \left[X(t, y(\alpha), \alpha) \right] = ?}$$

$$= [D_y X][D_\alpha y] + [D_\alpha X]$$

C^1– Dependence of Solution in Initial Condition & Parameters

Let $F : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ be C^1– function.

$$F : (X, \alpha) \longrightarrow F(X, \alpha)$$

(M1)

$$\boxed{d_\alpha \left[X(t, y(\alpha), \alpha) \right] = \,?}$$

$$= [D_y X][D_\alpha y] + [D_\alpha X]$$

C^1- Dependence of Solution in Initial Condition & Parameters

Let $F: \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ be C^1- function.

$$F : (X, \alpha) \longrightarrow F(X, \alpha)$$

(M1)

$$\boxed{d_\alpha \left[X(t, y(\alpha), \alpha) \right] = ?}$$

$$= \underbrace{\left[D_y X \right]}_{[\Phi(t)]} \left[D_\alpha y \right] + \underbrace{\left[D_\alpha X \right]}_{[\Psi(t)]}$$

C^1- Dependence of Solution in Initial Condition & Parameters

Let $F : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ be C^1- function.

$$F : (X, \alpha) \longrightarrow F(X, \alpha)$$

Let $\boxed{\Sigma(t) = d_\alpha\big(X(t; y(\alpha), \alpha)\big)} = ?$ M2

$$\frac{d}{dt} X(t; y(\alpha), \alpha) = F\big(X(t, y(\alpha), \alpha)\big)$$

d_α

$$\frac{d}{dt}[d_\alpha X] = [D_X F][d_\alpha X] + [D_\alpha F]$$

C^1 – Dependence of Solution in Initial Condition & Parameters

Let $F: \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ be C^1- function.

$$F: (X,\alpha) \longrightarrow F(X,\alpha)$$

Let $\boxed{\Sigma(t) = d_\alpha\Big(X(t; y(\alpha)),\alpha\Big)} = ?$

M2

$$\frac{d}{dt} X(t; y(\alpha)),\alpha) = F\Big(X(t, y(\alpha)),\alpha\Big)$$

$d_\alpha \downarrow$

$$\frac{d}{dt}[d_\alpha X] = [D_X F][d_\alpha X] + [D_\alpha F]$$

C^1- Dependence of Solution in Initial Condition & Parameters

Let $F : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ be C^1- function.

$$F : (X, \alpha) \longrightarrow F(X, \alpha)$$

Let $\boxed{\Sigma(t) = d_\alpha\Big(X(t; y(\alpha), \alpha)\Big)}$ $= ?$ M2

Then $\Sigma(t)$ solves:

$$\boxed{\begin{aligned} &\frac{d}{dt}\left[\Sigma(t)\right] = \left[D_x F(X, \alpha)\right]\left[\Sigma(t)\right] + \left[D_\alpha F(X, \alpha)\right] \\ &X(0; y(\alpha), \alpha) = y(\alpha) \Rightarrow \left[\Sigma(0)\right] = \left[D_\alpha y(\alpha)\right] \end{aligned}}$$

C^1- Dependence of Solution in Initial Condition & Parameters

Let $F: \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ be C^1- function.

$$F: (X,\alpha) \longrightarrow F(X,\alpha)$$

Let $\boxed{\Sigma(t) = d_\alpha\big(X(t; y(\alpha), \alpha)\big)}$

Then $\Sigma(t)$ solves:

$$\boxed{\begin{aligned} &\frac{d}{dt}[\Sigma(t)] = [D_X F(X,\alpha)][\Sigma(t)] + [D_\alpha F(X,\alpha)] \\ &X(0; y(\alpha), \alpha) = y(\alpha) \Rightarrow [\Sigma(0)] = [D_\alpha y(\alpha)] \end{aligned}}$$

inhomog

C^1 – Dependence of Solution in Initial Condition & Parameters

Let $F : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$ be C^1 – function.

$$F : (X, \alpha) \longrightarrow F(X, \alpha)$$

Let $\boxed{\Sigma(t) = d_\alpha\Big(X(t; y(\alpha)), \alpha\Big)}$

Then $\Sigma(t)$ solves:

$$\boxed{\Sigma(t) = [\Phi(t)][D_\alpha y(\alpha)] + \int_0^t \Phi(t)\Phi(s)^{-1} D_\alpha F(X(s,\alpha),\alpha)\, ds}$$